Optimal control for econometric models

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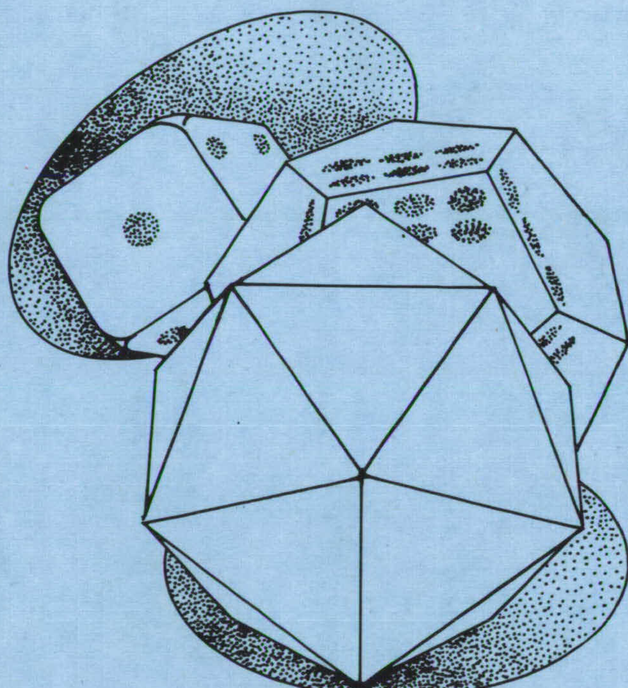
OPTIMAL CONTROL FOR ECONOMETRIC MODELS

An application of stochastic dynamic games



Max D. Merbis

STELLINGEN

behorende bij het proefschrift
OPTIMAL CONTROL FOR ECONOMETRIC MODELS
An application of stochastic dynamic games

I

De twee belangrijkste doelstellingen van het EG-landbouwbeleid, te weten zelfvoorziening en gegarandeerde inkomens voor de boeren, kunnen niet worden bereikt door het inzetten van slechts één instrument, namelijk het prijsbeleid (vgl. Tinbergens telregel, in Tinbergen (1956) of dit proefschrift, sectie 4.3).

J. Tinbergen (1956). Economic policy, principles and design, North-Holland, Amsterdam.

II

Het is beter te spreken van het certainty equivalence resultaat dan van het certainty equivalence principe. Hiermee wordt beoogd dat in situaties waar het resultaat geldt, de geldigheid inderdaad wordt aangetoond en dat in situaties waar het resultaat mogelijkerwijs niet geldt en als assumptie wordt gehanteerd, de gevolgen van deze werkwijze nader worden geanalyseerd.

III

Het is gewenst dat resultaten afkomstig van optimaal gestuurde econometrische modellen niet alleen de optimale paden van de doelvariabelen bevatten maar ook de bijbehorende betrouwbaarheidsintervallen en dat wordt nagegaan of de marges waarbinnen de doelvariabelen vallen politiek aanvaardbaar zijn.

IV

In Rhodes en Luenberger (1969) wordt een stochastisch dynamisch spel opgelost met behulp van de compensator techniek. In het geval dat beide spelers niet gedeelde observaties hebben, dient de oplossing gesuggereerd door de auteurs te worden vervangen door de oplossing gegeven in sectie 6.3.3 van dit proefschrift.

I.B. Rhodes, D.G. Luenberger (1969). Stochastic differential games with constrained state estimators, IEEE Trans. Automatic Control, vol. AC-14, pp. 476-481.

V

In de economische theorie die uitgaat van de hypothese dat agenten rationele verwachtingen hebben, wordt onvoldoende aandacht besteed aan de formulering van rationele verwachtingen in een econometrisch model, getuige de ad-hoc specificaties in bij voorbeeld Hughes Hallett en Rees (1983, p. 261) en Wallis (1980).

A.J. Hughes Hallett, H. Rees (1983). Quantitative economic policies and interactive planning, Cambridge University Press, Cambridge.
K.F. Wallis (1980). Econometric implications of the rational expectations hypothesis, Econometrica, vol. 48, pp. 49-73.

VI

Aan de ontwikkeling en uitwerking van technieken om economische systemen te regelen is en wordt veel aandacht besteed. Het is van belang dat de ontwikkeling van economische modellen waarvoor het zinvol is deze steeds meer verfijnde en geavanceerde technieken toe te passen, hiermee gelijke tred houdt.

VII

Bovenstaande stelling geldt met name voor de klasse van econometrische modellen waarin speltheoretische concepten een rol spelen.

VIII

Het is nuttig dat de onderzoeker die tot zijn werkterrein de toepassing van optimale besturingstheorie op economische modellen rekent, kennis neemt van de verworvenheden van de welvaartstheorie (zie bij voorbeeld Boadway en Bruce, 1984).

R.W. Boadway, N. Bruce (1984). Welfare economics, Basic Blackwell, Oxford.

IX

Kalman doet in zijn kritiek op de econometrische modelbouw (Kalman, 1983) de aanbeveling modellen te gebruiken die alleen uit de data informatie halen. Deze modelleringswijze heeft voor de economische praktijk nauwelijks waarde; zij biedt daarentegen goede mogelijkheden random generatoren uit te testen.

R.E. Kalman (1983). Identifiability and modeling in econometrics. In: P.R. Krishnaiah (ed.), Developments in Statistics, vol. 4, Academic Press, New York, pp. 97-136.

X

Veel sportblessures kunnen vermeden worden indien de fabrikanten van hardloopschoenen, in plaats van de als decoratie bedoelde stiksels en strepen, informatie verschaffen over de stijfheid van zool en contrefort, de vorm van het voetbed, het bij de schoen passende gewicht van de loper, en de ondergrond waarop hij (zij) kan lopen.

XI

De claim in Van den Herik (1983, p. 417) dat "De speelsterkte van schaakcomputers zal omstreeks het jaar 2000 hoger zijn dan de speelsterkte van de huidige wereldkampioen" is onjuist.

H.J. van den Herik (1983). Computerschaak, schaakwereld en kunstmatige intelligentie, Academic Service, Den Haag

M.D. Merbis, 11 april 1986.

OPTIMAL CONTROL FOR ECONOMETRIC MODELS

An application of stochastic dynamic games

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OPTIMAL CONTROL FOR ECONOMETRIC MODELS

An application of stochastic dynamic games

Max D. Merbis

Proefschrift ter verkrijging van de graad van doctor in de economische wetenschappen aan de Katholieke Hogeschool Tilburg, op gezag van de rector magnificus, prof. dr. R.A. de Moor, in het openbaar te verdedigen ten overstaan van een door het college van decanen aangewezen commissie in de aula van de Hogeschool op vrijdag 11 april 1986 te 16.15 uur door Maarten Dirk Merbis, geboren te Doetinchem.

Promotoren : Prof. dr. J. Wessels
Prof. dr. P.A. Verheyen

Assistent-promotoren: Dr. J.E.J. Plasmans
Ir. J.H. van Schuppen ph.d.

Contents page

Preface

The project from which this thesis originated is a common effort of Tilburg University and Eindhoven University of Technology. Participants of both institutions cooperate in the field of application and development of mathematical tools for economic models. The project could benefit from solid building stones: an econometric model formulated as a dynamic game was available, and algorithms for solving deterministic dynamic games were implemented. One of the remaining problems was the introduction of uncertainty into the model description. This problem will be tackled here.

A great number of people has contributed to the project. A special task force = Jaap Wessels and Joseph Plasmans (supervisors), Eric van Damme, Jan van Schuppen, Aart de Zeeuw, and at a later stage, Piet Verheyen = tried to keep the author on the(ir) right track. For their help and encouragement I am very grateful. Furthermore, I thank my former colleagues at Tilburg and Eindhoven for their support and advice. In particular I want to thank Toon van de Aker, who wrote some of the computer programmes, and Lex Meijdam, who assisted in the preparation of chapter 8.

After succeeding in finishing a fourth version of the manuscript the cumbersome burden to produce a final text was alleviated by
Lenie Spoor, who typed the manuscript,
Anja Meeder and Annemiek Dikmans, who assisted in typing,
Hildegard Penn, who improved the English text,
Jan Pijnenburg, who made the drawings,
Rob Alessie, who acted as a courier, and
Jeanette Hin, who made the cover design. She must be thanked for many other contributions as well.

Finally, the Common Research Pool of Tilburg University and Eindhoven University of Technology is acknowledged for providing the necessary funds.

Max D. Merbis
Amsterdam
February 4, 1986

Notation

I, I_n: unit matrix, of dimension $n \times n$

0, $0_{n\times m}$: matrix, of dimension $n \times m$, whose elements are zeroes

A_{i*}, A_{j*}: ith row, jth column of matrix A

$(A)_{ij}$: (i,j)th element of matrix A

$(A_1\ ,\ A_2)$, $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$: partitionings of matrix A

$(A_1;\ A_2) := \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$

$\mathrm{diag}(A_1,\ A_2) := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$

A^T, A^{-1}: transpose and inverse of A

rank(A), trace(A): rank and trace of A

$A > 0$, $A \geq 0$:A is a positive-definite, semi-positive-definite matrix, respectively

$\|x\|_Q^2 := x^TQx$

R, R^n: the real line, all n-tuples of real numbers

$F \vee G$: the smallest sigma-algebra generated by the sigma-algebras F and G

Note that the following notation is used for the n-dimensional row vector x^T: $x^T = (x_1;\ x_2;\ \ldots\ ;\ x_n)$.

CHAPTER ONE

INTRODUCTION

1.1. The problem field

The practical problem field from which the motivation for the subject of this thesis was derived is one of economic planning. In an economic planning problem an economic agent attempts to steer the state of the economy towards a more desired state by means of instruments available to him. The comprehensive volumes of Johansen (1977) contain precise definitions and discussions on the social and political need for results of economic planning. In this book we will turn attention to the quantitative problem of economic planning: formalized models of the economy will be used for planning. Characteristic factors of the economic process can be described by such models; in particular we will assume that several economic agents affect the economic process and that they have (partly) conflicting interests. Furthermore, the uncertainty, omnipresent in economic processes, will be taken into account. The precise role that uncertainty plays in the economic process and the way it is represented in the model will be one of the topics to be discussed and analysed later on. The central theme in this thesis is to take account of these two factors together. That is to say, we will consider a quantitative economic planning problem, in which several economic agents face an uncertain world. Although research has been done on subproblems, the combination of these aspects in an analytic model is relatively unexplored. The results obtained in the field of quantitative economic planning, will be a useful starting point and guideline to tackle our specific problem. A summary of the results, relevant to our study, will be given in the historical survey of quantitative economic planning (see section 1.3).

1.2. Elements of quantitative economic planning

In this section we will specify the essential elements of the quantitative approach to the economic planning problem. This will put the multi-agent setting under uncertainty in a more elaborate framework.

First, dynamic models of the economic process will be considered; in particular econometric models will be used. A second dynamic aspect is that the planning period extends over several time steps (a discrete-time setting is assumed throughout this book).

Secondly, the interaction between the agents is formalized by game-theoretical concepts (see Luce and Raiffa (1957), and Owen (1982) for introductions to game theory). This choice implies that the interaction between the agents refers to abstractions as competition, cooperation or hierarchy.

Thirdly, the uncertainty is represented by means of stochastic disturbances in the model; in particular, the model is affected by additive and/or multiplicative stochastic variables.

Finally, the plans of the agents will be derived by means of results from optimal control theory. This choice implies that the preferences of the agents are made explicit and maximized subject to the constraint of an econometric model.

The particular choices made above imply that the quantitative economic planning problem is formulated abstractly as a control problem for stochastic dynamic games; this explains the title of the book. A motivation for this approach follows. It is believed that an important class of economic processes fits the framework of stochastic dynamic games. Within this framework we are able to emphasize features of the process, like the multi-agent setting and uncertainty. Furthermore, the quantitative approach enables us to analyse the planning problem through mathematical theories. Using results from optimal control theory, the planning problem can be solved explicitly (at least in a number of cases), the solution can be implemented and application to real-world economic processes is feasible. The control approach has been applied successfully at several earlier occasions (see references in section 1.3).

In conclusion we can state that the goal of this thesis is to investigate whether the application of stochastic dynamic games is useful and promising for the economic planning problem. The procedure that will be followed to perform an analysis of quantitative economic planning within the proposed framework consists of three steps.

1. The formulation of a stochastic dynamic game as a suitable model for an economic process characterized by the multi-agent setting and decision making under uncertainty.
2. The formulation, analysis and analytic or numerical solution of the corresponding optimal control problem.
3. The algorithmic implementation and application of obtained results to econometric models of real-world economic processes. The evaluation of the results and feedback to the control approach to stochastic dynamic games.

1.3. A historical perspective of quantitative economic planning

We will present a short description of some relevant contributions in the field of quantitative economic planning. The historical development in this field will clarify the position of stochastic dynamic games.

Frisch and Tinbergen were the first to propose a distinction between target variables representing the explicit goals of economic policy, and instrument variables representing the available means through which economic agents pursue these goals. They assumed that a set of linear economic relationships expresses the relevant implications of movements in the instrument variables for movements in the target variables. In Tinbergen (1956) an equal number of target and instrument variables was assumed. In that case, the desired values of the target variables, specified by the agent, could be reached by a suitable choice of the instrument variables.

In Theil's formulation (Theil, 1964) the restriction of an equal number of target and instrument variables was dropped, and a preference function was introduced by which the outcomes of the decisions of an agent were ranked. In the optimum choice for the instrument variables, a situation could arise in which some or all target variables differ from their desired values. The relation between the target variables and the instrument variables was still a linear one; in addition, Theil analysed this problem in the context of decision making under uncertainty (uncertainty modelled as additive stochastic disturbances).

In later works, the basic distinction between target and instrument variables has been retained. The relationship between target variables and instrument variables, however, is represented by more sophisticated models. In Aoki (1976) and Chow (1975) the use of dynamic models has been advocated (see also Murata, 1982). Nonlinear models have been considered in Friedman (1975) and Garbade (1975). An advanced study using stochastic, dynamic models has been performed in Kendrick (1981).

These, mostly theoretical, findings have been used in various kinds of application studies. Many examples on the applicability of mathematical optimization methods to economic planning and case studies can be found in the economic journals, e.g. The Annals of Economic and Social Measurements, and The Journal of Economic Dynamics and Control. Large-scale applications have been performed by research teams in projects like PREM (Programme of Research into Econometric Methods, Imperial College, London) or the Cambridge Growth Project. Several books are devoted to the applications and refinements of optimization methods to economic systems, for instance Friedman (1975), Garbade (1975), Pindyck (1973), Pitchford and Turnovsky (1977).

More recently, there is a growing interest in models with an explicit account of the multi-agent setting and the interaction between the agents. Pindyck (1977), Haas (1981) and De Zeeuw (1984) are examples of theoretical studies with an empirical verification by means of multi-agent models. These authors have concentrated on deterministic, dynamic models.

From this partial list, we may conclude that the economic decision model has evolved from a static, linear relationship between target and instrument variables, towards a dynamic model that incorporates stochastic and multi-agent aspects. This progress has been possible, because the corresponding dynamic, stochastic optimization problems can be analysed by means of recent progress made in mathematical programming and in systems and control. However, relatively little attention has been paid to stochastic dynamic models with more than one economic agent. The application of stochastic dynamic games to quantitative economic planning seems to be one of the next steps to be taken in this field.

1.4. Outline of the book

From the discussion in the previous sections three kinds of issues can be deduced: the modelling, the optimization and the implementation problem.

These topics are strongly related: we must confine our attention to specific classes of mathematical models for the formulation of stochastic dynamic games in order to solve the optimization problem, and in order to verify the applicability to real-world case studies. Typical examples as to how to fulfil these limitations are provided by the literature on quantitative economic planning (section 1.3).

In chapters 2, 3 and 4 aspects of modelling will be discussed; in chapters 5, 6 and 7 various kinds of optimization problems will be analysed; in chapter 8 an application study will be performed. In some detail the contents of the chapters are as follows:

In chapter 2 optimal control theory will be reviewed. We shall define a class of econometric models to be used in this book and claim that the results of optimal control are applicable to this type of econometric model. In chapter 3 the definitions that are required to specify a stochastic dynamic game and a problem formulation will be presented. In chapter 4 a stochastic dynamic game will be analysed with respect to the information available to and the preference function stated by the economic agent. A classification of various types of dynamic optimization problems results based upon the information available to the agents. In chapters 5, 6 and 7 three such types will be analysed. For the optimal control problem discussed in chapter 5 an analytic and tractable solution to the planning problem can be given. In chapter 8 this case will be applied to a linked macroeconometric policy model with two decision makers.

CHAPTER TWO

THE CONTROL APPROACH

2.1. Introduction

In this chapter we will discuss the basic concepts of and results from optimal control theory and show its relation to econometric models. This will be done in a number of steps. First, the basic scheme that underlies the control approach will be formulated, a scheme which allows a general form for the model and for the preference function of the economic agent. The use of this scheme enables us to emphasize the fundamental notions of the control approach and to confront it to alternative approaches. Secondly, we will discuss how the elements in this scheme can be substantiated, when we deal with econometric models, in a multi-agent setting and under uncertainty. Thirdly, we will briefly review standard results in optimal control theory. Finally, we will show that these results of optimal control theory are applicable to the type of econometric model under consideration.

2.2. The control-theoretic scheme

In this section we present a stylized scheme which summarizes the basic elements of the control approach. It is introduced here to show the essential mechanism of the control method, and to confront it with other views on the planning problem. In addition, it displays the various aspects of the role that the economic agent must fulfil.

The control-theoretic scheme

1. The economic agent is able to manipulate or control the economic system. To that aim the agent employs so-called instruments, which are formalized as the instrument variables u_{con}.
2. Other variables beyond the control of the economic agent influence the economic system as well. These variables are called the uncon-

trollable exogenous variables (called "data variables" by Tinbergen) and denoted by u_{unc}.

3. The economic goals of the economic agent are called the targets, which are formalized as target variables z.
4. The relation between the targets and the instruments is condensed into the decision model of the economic system. This relation is represented here by $z = f(u_{con}, u_{unc})$; this means that values for the target variables result if we insert values for u_{con} and u_{unc} into the function f.
5. The economic agent is supposed to rank his decisions in, what is usually called in economics, a preference or welfare function. In control theory this is called a cost function, and the notation $J(u_{con})$ will be used.

It is the task of the economic agent to specify or determine the sets U_{con}, U_{unc} and Z of all admissible u_{con}, u_{unc} and z, respectively, the cost function J and the function f.

The following steps characterize the control approach to economic planning.

1. Predict u_{unc}; let the result be $\bar{u}_{unc} \in U_{unc}$.
2. For $z = f(u_{con}, \bar{u}_{unc})$, search through all $u_{con} \in U_{con}$ so as to find the one (one of these) which yields the lowest value for $J(u_{con})$.

The control-theoretic scheme underlies the planning methods as advocated by the contributors in this field (see section 1.3). We have left out the time aspect, the stochastic aspect and the multi-agent aspects here. These aspects can be introduced in making specific choices for the elements of this scheme.

We want to digress here, and mention two examples of planning methods which differ from the control-theoretic scheme.

First, suppose that the economic agent does not attempt to state $J(u_{con})$ explicitly. Suppose he predicts u_{con}, say by $\bar{u}_{con} \in U_{con}$, in a similar way as done for u_{unc}. The predicted target variables follow from $\bar{z} = f(\bar{u}_{con}, \bar{u}_{unc})$. The triple $(\bar{z}, \bar{u}_{con}, \bar{u}_{unc})$ is called a scenario. A series of different scenarios, based on several choices for $\bar{u}_{con}$, may provide insight into the behaviour of the economic system and its development under the instrument variable u_{con}. This approach is frequently used for policy evaluation by macroeconomic models. The second step in the control approach, the optimization step, has been replaced by a simulation study.

Secondly, we remark that an assumption inherent in the scheme is that the economic agent behaves rationally, i.e. he acts such that he maximizes his preferences. Although this stance is almost universally accepted in economic theory, it may be more realistic in the explanation of the actual behaviour of the economic agent that he does not optimize, but satisfice his behaviour. In economic literature examples can be found of this approach; Cyert and March developed a model of the firm, including multi-agent and stochastic aspects. For a criticism and summary of this model, as well as a definition of the notion of satisficing behaviour, see Koutsoyiannis, 1979, chapter 18.

2.3. Limitations of, and choices for the control approach

Let us consider again the control-theoretic scheme. The elements in this scheme have not yet been specified. We will proceed by providing arguments how this can be done. The following items must be discussed, as is apparent from the scheme. First, a class of mathematical models must be specified. Such a model replaces the formal relation $z = f(u_{unc}, u_{con})$ and incorporates the multi-agent setting, uncertainty and dynamics. Secondly, we notice that an economic agent will base his decisions on information available to him. It must be specified in the model how the relation between information and control actions is substantiated, for each time t. Thirdly, in order to obtain the actual solution of the optimal control problem, we require a specific form for the cost function. The preferences of the economic agent are to be expressed in this form.

When a particular choice has been made, the proposed framework for a stochastic dynamic game may, or may not, lead to tractable algorithms generating optimal controls. We will attempt to make choices that are most likely to lead to tractable results. Concerning the class of mathematical models, we shall deal with econometric models in structural form (see Intriligator, 1978). This appears to be a frequent choice within the theory of quantitative economic planning. Arguments in favour of this choice are: an extensive estimation methodology is available; the multi-agent setting is feasible; uncertainty is incorporated, being the random shocks that affect the evolution of the economic process. Concerning the notion of information and the form of the cost function, we will follow the standard approach of optimal control theory. Tractable results are available for the so-called linear-quadratic Gaussian (LQG) control problem. This means that we use a linear system, a quadratic cost function and Gaussian stochastic disturbances. It will turn out that this theory is applicable to econometric models in structural form, and that it is subject to economic interpretation. The specification of available information and the form of the cost function follow immediately, when we focus attention on LQG-models.

In the next three sections we will display the choices indicated above. In section 2.4 we will present an econometric model in structural form; in section 2.5 the LQG-problem will be introduced and in section 2.6 the connection between LQG-theory and econometric models will be established.

2.4. The econometric model

In this section we will establish the relation between the instrument variables and the target variables, using an econometric model in a so-called structural form. Notation will be introduced before we present its definition.

Let (Ω,F,P) be a complete probability space, and $T = \{0,1,\ldots\}$ the time-index set.

Define

$u: T \to R^m$, the exogenous variables
$z: \Omega \times T \to R^r$, the target variables
$y: \Omega \times T \to R^k$, the endogenous variables
$v: \Omega \times T \to R^\ell$, the disturbance process.

The exogenous variables u consist of instrument variables (u_{con}) and uncontrollable exogenous variables (u_{unc}). For simplicity of notation we do not make this distinction until the actual control problem is solved (this will be done in chapter 5). The target variables z are economic variables which express the objectives of the decision maker. The relation between z and u will be established later through the introduction of the endogenous variables y. This set-up allows the model designer quite some flexibility. The notation $v(t) \in G(0,V)$ will be used, when the disturbance process $(v(t), t \in T)$ has a Gaussian distribution with zero mean and variance $V = V^T \in R^{\ell\times\ell}$. Furthermore, we say that $(v(t), t \in T)$ is white, if $E[v(t)\, v(s)^T] = 0,\ t \neq s,\ t, s \in T$.

An econometric model in structural form represents a certain class of mathematical models indicated as ARX(p,q)-models, or simply as ARX-models. This abbreviation stands for an autoregressive model of order p with exogenous input of order q, driven by white noise. Formally we have the relation

$$
\begin{aligned}
y(t) = {} & A_0 y(t) + A_1 y(t-1) + \dots + A_p y(t-p) + \\
& B_0 u(t) + B_1 u(t-1) + \dots + B_q u(t-q) + Mv(t)
\end{aligned}
\tag{2.1}
$$

with $v(t) \in G(0,V)$, $(v(t), t \in T)$ is white.

The target variables z can be expressed as a linear combination of some of the endogenous and exogenous variables.
We assume that

$$
z(t) = Hy(t) + Ju(t) \tag{2.2}
$$

We observe that (2.1) and (2.2) establish a linear relation between u and z, for which the following properties hold (as claimed in section 2.3). The uncertainty is present through the "random shock" $Mv(t)$, representing the uncertain evolution of the economic system. The multi-agent set-up is immediate because y, u and z can be composed of endogenous, exogenous and target variables, attributed to several economic agents. A large class of estimation methods is available to estimate the unknown parameters of (2.1), e.g. Theil, 1971; Maddala, 1979; Intriligator, 1978.

2.5. Optimal control theory: the LQG-problem

In this section we will formulate the optimal control problem for a class of mathematical models known as Gaussian systems. We will specialize to the case for which analytic results are available. This case has proved to be fruitful in many practical applications.

We will start by presenting a representation of a Gaussian system. The interpretation of the variables to be used in this representation is independent of the one given in the previous section, although the same symbols may have been used. The connection between the variables in both interpretations will be given later.

A Gaussian system will be represented as follows

$$x(t+1) = Ax(t) + Bu(t) + Mv(t),\ x(0) \tag{2.3a}$$

$$y(t) = Cx(t) + Du(t) + Nv(t) \tag{2.3b}$$

$x(0) \in G(m_0, \Sigma_0)$, $T = \{0,1,...\}$ the time-index set

$v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$.

The variables involved in (2.3) have the following, not necessarily economic, interpretation. $x: \Omega \times T \to R^n$ is the state process; the state $x(t)$ may be interpreted as the memory of the system. $u: T \to R^m$ is the input variable or the control variable, which can be employed by the decision maker.*) The output process $y: \Omega \times T \to R^k$ is used to obtain information about the state of the system. It represents the measurements by which the decision maker observes the state. $v: \Omega \times T \to R^\ell$ is the disturbance process, assumed to be Gaussian and white, as in (2.1). $Mv(t)$ and $Nv(t)$ represent system and measurement noise respectively. If $MVN^T = 0$, these noise processes are uncorrelated. $x(0)$ and $(v(t),\ t \in T)$ are the basic random variables of (2.3); the distribution of x and y can be derived from the known distribution of the basic random variables. In

*) The expressions decision maker and economic agent are used as equivalents.

addition, we introduce the controlled variable $z: \Omega \times T \rightarrow R^r$. The decision maker is supposed to steer or manipulate the controlled variable z, by means of his input u, while he observes the system's state through y. We will assume that

$$z(t) = Hx(t) + Ju(t) \tag{2.4}$$

This definition for z equals the one given in Kwakernaak and Sivan, 1972, p. 476. In most other sources in the literature, however, it is assumed that J = 0. The present form (2.4) will facilitate the transformation of the econometric model to the Gaussian system representation, which will be presented later on.

In this set-up, the following problem may be considered. Suppose a reference path is given for the controlled variable: $(\bar{z}(t),\ t \in [t_0, t_f])$ over a given time interval $[t_0, t_f] \subseteqq T$. Then the regulator problem for (2.3) roughly is the following. For a given path $(\bar{z}(t),\ t \in [t_0, t_f])$ find an appropriate input, so that the controlled variable tracks the reference path. Often a practical constraint is encountered. The range of values over which the input u(t) is allowed to vary, is limited. We will deal with this problem when we state the optimal control problem (as a regulator problem) rigorously. Before we can do this, two more concepts are required: the information pattern and the cost function.

Information pattern

It must be specified which information the decision maker has available at each time t and which information he uses to base his control on. For example, for the Gaussian system representation (2.3), a natural choice for the information available to the decision maker at time t is the set of all inputs and outputs up to and including time t (obviously, under the assumption that the decision maker has perfect recall).

With respect to the information that is used to determine the control, we will distinguish two cases: the complete-state observation case and the partial-state observation case (also called the perfect-state and the imperfect-state observation case, respectively).

In the complete-state observation case, the decision maker observes the complete state $x(t)$. Let the state $x(t)$ provide the information upon which the control $u(t)$ will be based. The relation between this information and the control will be established as follows. Suppose that for each $t \in T$ the control value is to be selected from a prespecified action set $A \subseteq R^m$, and that $x(t)$ takes its values from the state space $X \subseteq R^n$. An admissible feedback control law is then any sequence $g = \{g_0, g_1, \ldots\}$ with g_t a function from X to A, such that $u(t) = g_t(x(t)) \in A$ for all $x(t) \in X$. In this particular case, g is called a feedback control law or feedback policy; the class of all admissible feedback control laws will be denoted by $\underline{U}$.

In the partial-state observation case, we proceed analogously. Again $A \subseteq R^m$ is the action space, and $Y \subseteq R^k$ is the output space. The decision maker is supposed to use all past observations for his control. Now $g = \{g_0, g_1, \ldots\}$ is a closed-loop control law with $g_t: Y^t \to A$, such that $u(t) = g_t(y(0), y(1), \ldots, y(t-1)) \in A$. Again this class of all admissible closed-loop control laws will be denoted by $\underline{U}$. The distinction from the class of feedback laws will be clear from the context.

One may ask why feedback control laws in the complete-state observation case do not incorporate all past states, as in the case of the closed-loop control laws. The reason is that we consider Markovian systems. Let $p(\cdot|\cdot)$ denote a conditional distribution, then one can prove that (2.3) with $u(t) = g_t(x(t))$ possesses the Markov property

$$p(x(t+1)|x(t), x(t-1), \ldots, x(0)) = p(x(t+1) \mid x(t)),$$

see Jazwinski, 1970, section 3.9. The Markov property implies that, if $x(t)$ is known, knowledge about $x(t-1), \ldots, x(0)$ is redundant. This property holds because $(v(t),\ t \in T)$ is white and independent of $x(0)$. It justifies the use of feedback control laws.

The difference between control actions $u(t)$ and control laws g must be emphasized: the action is the outcome of the mapping, the control law is the mapping itself. For clarity of exposition, different symbols u and g have been used here; frequently, in control theory, this distinction in notation is discarded and the symbol u is used for both objects. In the next chapters we will adopt this convention.

In the examples above, the control law maps the state or the past observations into the action space. We can generalize by considering an information pattern that specifies the information at time $t \in T$ which the decision maker is supposed to use for his control $u(t)$. Let us denote the specified information by $\eta(t)$, then we have $u(t) = g_t(\eta(t))$. In the case of feedback laws, we let $\eta(t) = x(t)$; in the case of closed-loop laws, we let $\eta(t) = (y(0),\ldots,y(t-1))$.

When the control $u(t)$ does not depend on $\eta(t)$, we deal with open-loop control. The control law is a deterministic function $u: T \to A$, specified by the decision maker at the start of the control period. He will apply this sequence of actions to the control system, irrespective of the evolution of the state or the observations he may make. In this book we will only consider feedback and closed-loop control laws. A comparison between open-loop and feedback control is discussed extensively in the literature, see a.o. Kwakernaak and Sivan (1972), Åström (1970), Callier and Desoer (1982).

The cost function

The cost function for the regulator problem is formulated as follows. The decision maker wishes to steer the controlled variable towards the reference path $(\bar{z}(t),\ t \in [t_0,\ t_f])$. This path can be approached by the use of the control variable. Since the range of values over which $u(t)$ is allowed to vary, is limited, we assume that the decision maker states a reference path $(\bar{u}(t),\ t \in [t_0,t_f])$ for the control variable as well. Deviations from these desired paths will be penalized, stagewise, in the following way.

Let g be an admissible control law and $\underline{U}$ the class of admissible control laws. The costs, and the expected or average costs, associated with g, are, respectively

$$C^g := \sum_{t=t_0}^{t_f} \{\|z(t)-\bar{z}(t)\|_Q^2 + \|u(t)-\bar{u}(t)\|_R^2\} \tag{2.5a}$$

$$J(g) := E[C^g] \tag{2.5b}$$

The notation $\|x\|_Q^2 = x^T Qx$, $Q = Q^T$ has been used.

An explanation of (2.5) follows. Through the introduction of g, the control u(t) becomes a stochastic process by $u(t) = g_t(\eta(t))$; the random variables x, z and y in (2.3) are well defined and depend on g. Their probability distribution follows from the choice of g and the distribution of the basic random variables. C^g is a stochastic variable and in general no g will exist which minimizes the costs uniformly for all sample points $\omega \in \Omega$. This difficulty is bypassed by considering the average of expected costs $J(g) = E[C^g]$, where E denotes expectation with respect to the basic random variables.

Optimality is defined in terms of expected costs. The feedback control law $g^* \in \underline{U}$ is called optimal if

$$J(g^*) = J^* = \inf\{J(g) \mid g \in \underline{U}\} .$$

The optimal control problem

We will state the optimal control problem for the partial-state and the complete-state observation case.

The partial-state observation case

The LQG-control problem is:

$$\min_{g \in \underline{U}} \text{ minimize } J(g) \text{ subject to}$$

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t) \\ y(t) &= Cx(t) + Du(t) + Nv(t) \\ z(t) &= Hx(t) + Ju(t) \end{aligned}$$

g denotes the closed-loop control law, such that $g = \{g_0,\dots,g_{t_f}\}$, $g_t = g_t(y(0),\dots,y(t-1))$; the corresponding class of admissible control laws is denoted by $\underline{U}$. J(g) is given in (2.5).

The complete-state observation case

The LQG-problem is:

$$\underset{g \in \underline{U}}{\text{minimize}}\ J(g) \text{ subject to}$$

$$\begin{aligned} x(t+1) &= A\, x(t) + B\, u(t) + M\, v(t) \\ z(t) &= H\, x(t) + J\, u(t) \end{aligned} \tag{2.6}$$

g denotes the feedback control law, such that $g = \{g_0, \ldots, g_{t_f}\}$, $g_t = g_t(x(t))$; the corresponding class of admissible control laws is denoted by $\underline{U}$. J(g) is given in (2.5).

The formulation of the LQG-problem can be simplified when we substitute the expression for z into J(g). The result is a standard optimal control problem, in terms of x, y and g. (See Bertsekas, 1976; Åström, 1970).

The advantages of this set-up are many. In the complete-state observation case it has been proved that, under the conditions $Q \geq 0$, $R > 0$ and $A = R^m$ (no restrictions on the action space), the optimal control $u^*(t)$ is a linear feedback rule in x(t) and is unique. Similar results hold for the partial-state observation case, see Bertsekas (1976), Kwakernaak and Sivan (1972). The behaviour of (2.3) under the feedback control $u^*(t)$, as well as under arbitrary control u(t), has been analysed with respect to aspects as stabilizability, sensitivity to parameter changes and disturbances, see Safonov (1980), Callier and Desoer (1982), Frank (1978). Besides extensive theoretical investigations, the solution to the linear-quadratic Gaussian control problem has been applied to many technical models (see the journals IEEE Transactions on Automatic Control and Automatica J. IFAC).

2.6. Optimal control for econometric models: a synthesis

In this section we will confront the results of the linear-quadratic Gaussian control problem to an econometric model of ARX(p,q)-type. Special attention will be paid to the interpretation of the economic variables in a control-theoretic setting.

The main step is the transformation of the ARX(p,q)-model to a state-space form. In section 5.2 we shall prove that the ARX(p,q)-model (2.1) can be transformed into a Gaussian system representation of the form

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + \bar{M}v(t+1) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.7}$$

where x(t) is defined as a vector consisting of, possibly delayed, endogenous and exogenous variables. The matrices A, B, C, D in (2.7) can be derived from $A_0,\ldots,A_p$, $B_0,\ldots,B_q$ in (2.1); the matrix $\bar{M}$ in (2.7) follows from M in (2.1). Together with (2.2), we have for (2.7)

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + \bar{M}v(t+1) \\ z(t) &= HCx(t) + (J+HD)u(t) \end{aligned} \tag{2.8}$$

The representation obtained in (2.8) is essentially identical to the one of (2.6), because $(v(t),\ t \in T)$ is white noise. Notice that we are in the complete-state observation case: the state x consists of observed values of past and present endogenous and exogenous variables.

The following interpretation will be given to the elements of an econometric model. The ARX(p,q)-model is transformed to a state-space representation; the target variables z serve as the controlled variables, for which a desired target path $(\bar{z}(t),\ t \in [t_0,t_f])$ must be specified. The instrument variables serve as control variables; when solving the control problem, the economic agent is supposed to divide the exogenous variables u(t) into controllable variables (the instrument variables) and uncontrollable exogenous variables. The economic agent is supposed to specify a desired or anticipated path for the exogenous variables $(\bar{u}(t),\ t \in [t_0,t_f])$. In this way, the form of the preference function of the economic agent completely parallels the form of the cost function of the decision maker. Finally, the economic agent is supposed to use past and present values of endogenous and exogenous variables, subsumed in the state x(t), for his control. This justifies the use of feedback laws.

Many problems in economics can and have been tackled via the control approach. The outstanding achievements of the linear-quadratic Gaussian control problem have stimulated its use in economic applications. It must be emphasized that these properties are mainly due to the fact that we deal with the complete-state observation case, the single-decision-maker case, a linear model, a quadratic cost function and Gaussian white noise processes. If we relax these assumptions and consider the partial-state observation case, the multi-agent case, nonlinear models, general cost functions and disturbances, things are becoming complicated.

We will limit attention to the area of linear systems, quadratic cost functions and Gaussian disturbances. However, we will consider the partial-state observation case and the multi-agent case in the next chapters.

CHAPTER THREE

PROBLEM FORMULATION AND DEFINITIONS

3.1. Introduction

In this chapter we will consider explicitly the multi-decision-maker set-up in quantitative economic planning. We will provide definitions required for the specification of stochastic dynamic games, and present a problem formulation for the control approach to stochastic dynamic games which consists of three parts.

First, the specification of a stochastic dynamic game as an appropriate model for economic processes will be considered. The central issue in our treatment is information. This seems most convenient, because the complexity of the problem of specification arises from the fact that different decision makers may have different information.

Secondly, we consider the formulation of the corresponding control problem. Obviously, the main objective is to solve this problem, preferably in such a way that tractable algorithms result.

Finally, the control solutions will be used for implementation and application to real-world economic models. Criteria must be developed to evaluate the usefulness of the control approach to stochastic dynamic games.

The three topics listed above are the contents of this chapter; the specification of a stochastic dynamic game will be treated in section 3.2, the formulation of the control problem in section 3.3 and criteria for evaluation of the control approach in section 3.4.

3.2. Specification of a stochastic dynamic game

In this section we will describe the elements that characterize a stochastic dynamic game. For simplicity of notation and for convenience, we will consider models with two decision makers. In most of the cases treated in this book, the generalization to the multi-decision-maker case is straightforward. Control problems for models with two decision makers are much more complex than for a single-decision-maker model. A major reason is that different decision makers may have different information which they use in generating their decisions. Therefore it is felt that an extensive and detailed problem formulation is required. Aspects of information in the specification of a stochastic dynamic game will play a central role (section 3.2.1). The specification will be completed by treating concepts for the interaction between the decision makers and the form of the cost function (section 3.2.2).

In this chapter we will give definitions only of those elements that constitute a stochastic dynamic game. The analysis that shows how a consistent and appropriate choice for these elements must be made from an econometric modelling point of view is given in the next chapter.

3.2.1. The Gaussian system representation

It is appropriate here to review the procedure of constructing a mathematical model from input-output data. The result of this procedure will enable us to define the notion of information consistently with the underlying modelling procedure. Some notation and terminology will be introduced first.

Denote by E the economic process (or shortly: the economy), the behaviour of which we are interested in studying, modelling and controlling. The economy is supposed to have a measurable input signal u and to generate an output signal y, see figure 3.1. The set $\{u(s), y(s), s = 0,1,...\}$ will be called the input-output data or the on-line model data.

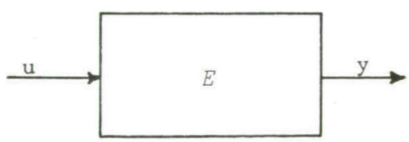


Figure 3.1. Economy E

An econometric model will be constructed which represents the properties of the economy relevant to the purposes of the model builder.

This construction is supposed to be done by an identification procedure, that consists of the following steps

1. choose a class of mathematical models
2. select a particular member of this class that explains the output data from the input data.

In the first, and crucial, step a class of mathematical models is chosen; knowledge from economic theory may be incorporated in the structure of this class. The second step is performed by a parameter estimation procedure. Usually a third step in which the outcome is evaluated completes the identification procedure (see Goodwin and Payne, 1977; Box and Jenkins, 1974; Ljung and Söderström, 1983).

The following notation will be used to formalize the procedure outlined above. Let $M(\theta)$ be a particular model, where θ represents the parameters of the model (usually a vector). θ is allowed to range over Θ, a subset of R^d (d is the dimension of the parameter vector). As θ ranges over Θ, the model set $\underline{M} := \{M(\theta) | \theta \in \Theta\}$ will be obtained. Examples will clarify this notation.

Example 3.1

a. The ARX(p,q)-model, see (2.1), with $k = 1$, $m = 1$, $\ell = 1$, $p = 1$, $q = 1$, obeys

$$y(t) = a_0 y(t) + a_1 y(t-1) + b_0 u(t) + b_1 u(t-1) + v(t)$$

with $v(t) \in G(0,V)$.
In this example we have $\theta = (a_0, a_1, b_0, b_1, V)$,
$\Theta = R^4 \times R_+ \ (R_+ := \{x \in R | x \geq 0\})$.

b. The Gaussian system representation

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t),\ x(0) \\ y(t) &= Cx(t) + Du(t) + Nv(t) \end{aligned}$$

with $v(t) \in G(0,V)$, $(v(t), t \in T)$ is white and independent of $x(0)$, $x(0) \in G\ (m_0, \Sigma_0)$.

In this example it is convenient to represent θ as a set instead of as a vector:

$$\theta = \{A, B, C, D, M, N, V, m_0, \Sigma_0\}$$

□

Until now no attempt has been made to incorporate a multi-agent set-up. This will be done using the concepts defined above. We will distinguish <u>structural</u> information, i.e. $M(\theta)$, and <u>observational</u> information, i.e. the input-output variables.

The following assumption will be made: the set $\underline{M}$ is taken fixed and identical for both agents. This assumption constitutes a major simplification to the case that both agents have separate model sets. Relaxation of this assumption will complicate the analysis to a large extent (cf. the corresponding remark in section 4.2.2, ad 3).

<u>Definition 3.2</u>

a. The <u>on-line model data for E</u> at time $t \in T$ consists of the set $I_t := \{y(s), u(s), s < t\}$.
b. The <u>information pattern of DMi</u> specifies which components of the variables in I_t are available to DMi. This set of variables is denoted by $\eta_t^{(i)}$, the on-line model data of DMi at time $t \in T$.

The on-line model data for E are the economic time series consisting of all endogenous and exogenous variables. The second part of the definition makes it possible that not all agents have access to all these variables. The information pattern of DMi specifies that DMi knows $\eta_t^{(i)} = \{y_i(s), u_i(s), s < t\}$, $i = 1,2$, where $y(s) = (y_1(s); y_2(s))$, $u(s) = (u_1(s); u_2(s))$.

<u>Definition 3.3</u>

a. The <u>off-line model data for E</u> is the set of parameter values θ in $M(\theta)$.
b. The <u>off-line model data of DMi</u> is the subset θ_i of θ available to DMi.

This definition parallels the previous one, but specifies the structural information. Part b. of the definition reflects that DMi's parameter set θ_i may be estimated from $\eta_t^{(i)}$, a strict subset of I_t.

Again we will clarify these definitions by an example. We will use a Gaussian system representation with two decision makers who both observe the system's state.

<u>Example 3.4</u>

Consider a Gaussian system representation for DM1 and DM2.

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u_1(t) + B_2u_2(t) + Mv(t),\ x(0) \\ y_1(t) &= C_1x(t) + D_1u_1(t) + N_1v(t) \\ y_2(t) &= C_2x(t) + D_2u_2(t) + N_2v(t) \end{aligned} \tag{3.1}$$

where

$v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$,

$x(0) \in G(m_0,\ \Sigma_0)$.

Let the on-line model data of DMi at time t be $\{u_i(s),\ y_i(s),\ s < t\}$, $i = 1,2$.

Now we suppose that

$$\eta_t^{(i)} = \{u_i(s),\ y_i(s),\ s < t\},\ i = 1,2$$

$$\theta_1 = \{A,B_1,B_2,C_1,D_1,M,N_1,m_0,\Sigma_0,V\}$$

$$\theta_2 = \{A,B_1,B_2,C_2,D_2,M,N_2,m_0,\Sigma_0,V\}$$

Of course, many other specifications for $\eta_t^{(i)}$ and θ_i, $i = 1,2$, are possible, depending on the particular applications one has in mind. □

The information patterns for on-line model data and the off-line model data of the decision makers will specify the stochastic dynamic game, given $\underline{M}$. The interpretation of on- and off-line model data and the relation between them will be discussed in the next chapter. The following definition will be required.

Definition 3.5

The on-line model data of DM1 and DM2 are called shared if $\eta_t^{(1)} = \eta_t^{(2)}$ for all $t \in T$ and all possible realizatons, and are called non-shared otherwise. The off-line model data are called shared if $\theta_1 = \theta_2$, and non-shared otherwise.

□

We will conclude this section by presenting some examples of information patterns for on-line model data. We will restrict our attention to the special case in which the decision makers face a planning period $[t_0, t_f]$, using the model (3.1). For this case we can extend the notion of information pattern, as used in section 2.5 for feedback and closed-loop control laws, to the multi-agent case. Thus it is understood that DMi will base his control $u_i(t)$ on the information $\eta_t^{(i)}$. $\eta_t^{(i)}$ will only consist of values of output or state variables, not of control variables.

Similar definitions were proposed by Başar and Olsder, 1982, chapter 5.

Definition 3.6

An information pattern of DMi at time $t \in T$ is called

a. open-loop if $\eta_t^{(i)} = x(t_0)$
b. feedback perfect-state if $\eta_t^{(i)} = x(t)$
c. feedback imperfect-state if $\eta_t^{(i)} = y_i(t)$
d. closed-loop perfect-state if $\eta_t^{(i)} = (x(t_0),\dots,x(t))$
e. closed-loop imperfect-state if $\eta_t^{(i)} = (y_i(t_0),\dots,y_i(t))$
f. one-step delayed observation sharing (1sDOS) if
$$\eta_t^{(i)} = \{y_1(s),\ y_2(s),\ t_0 \leq s \leq t-1\} \cup y_i(t)$$

Notice that in the cases a, b and d the information is shared; in the cases c, e and f the information is non-shared. This does not imply, however, that $y_1(t)$ and $y_2(t)$ are necessarily uncorrelated, see (3.1).

3.2.2. Cost functions and solution concepts

The cost functions for the decision makers are stated analogous to (2.5): DMi is supposed to minimize his expected or average costs, given by

$$E\Big[\sum_{t=t_0}^{t_f}\{\|z_i(t)-\bar{z}_i(t)\|^2_{Q_i}+\|u_i(t)-\bar{u}_i(t)\|^2_{R_i}\}\Big]\ , \qquad i = 1,2 \tag{3.2}$$

Note that DMi's expected costs depend on $u_j(t)$, $i \neq j$, only via the dependence of z_i on x. The LQG-problem can still be solved when (3.2) incorporates a term $\|u_j(t)-\bar{u}_j(t)\|^2_{R_{ij}}$, $i \neq j$, $i,j = 1,2$.

In a multi-agent approach it does not suffice to state the cost functions, but also a particular solution concept must be chosen. A solution concept is a normative concept which formalizes the real-world interaction between the decision makers into abstractions like cooperation, competition or hierarchy.

Before we list the definitions of various solution concepts, we will explore the relation between the cost functions of the decision makers. A general setting will be used. So, let $\underline{U}_i$ be the class of admissible control laws of DMi and J_i: $\underline{U}_1 \times \underline{U}_2 \to R$ the cost function of DMi, i = 1,2. Then we can classify the cost functions into three cases.

1. $J_1(u_1,u_2) + J_2(u_1,u_2) = 0$ for all $(u_1,u_2) \in \underline{U}_1 \times \underline{U}_2$. The decision makers are antagonists: the gain of one of them is the loss of the other. This is called the zero-sum case.
2. $J_1(u_1,u_2) + J_2(u_1,u_2) \neq 0$ for at least one (u_1,u_2) in $\underline{U}_1 \times \underline{U}_2$. The decision makers are supposed to have partially common objectives, partially opposed interests. This is called the nonzero-sum case.
3. $J_1(u_1,u_2) = J_2(u_1,u_2)$ for all $(u_1,u_2) \in \underline{U}_1 \times \underline{U}_2$. Both decision makers have common objectives. If they have also shared information patterns for on- and off-line model data, the game reduces to the single-decision-maker case.

Depending on the particular situation to be modelled, a choice for the relation between the cost functions must be made. In addition, a cooperative, competitive or hierarchical solution concept must be chosen. The game-theoretical literature (Luce and Raiffa, 1957; Owen, 1982) provides a list of solution concepts. The most important concepts are reviewed here, for the two-decision-makers case. As mentioned above, we will use the strategy sets $\underline{U}_1$ and $\underline{U}_2$ and the cost functions J_1 and J_2.

Definition 3.7

a. The zero-sum case. Let $J := J_1 = -J_2$.
(u_1^*, u_2^*) is called a saddle point, if

$$J(u_1^*, u_2) \leq J(u_1^*, u_2^*) \leq J(u_1, u_2^*) .$$

The former inequality holds for all $u_2 \in \underline{U}_2$, the latter for all $u_1 \in \underline{U}_1$.

b. The nonzero-sum case. Let $J_1 \neq J_2$.

1. (u_1^*, u_2^*) is called a Nash equilibrium (or: Nash) pair, if

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) \qquad \text{for all } u_1 \in \underline{U}_1$$

$$J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2) \qquad \text{for all } u_2 \in \underline{U}_2$$

2. (u_1^*, u_2^*) is called a Pareto-efficient (or: Pareto) solution, if for all $(u_1, u_2) \in \underline{U}_1 \times \underline{U}_2$, with $(u_1, u_2) \neq (u_1^*, u_2^*)$, either

$$J_1(u_1^*, u_2^*) = J_1(u_1, u_2)$$

$$J_2(u_1^*, u_2^*) = J_2(u_1, u_2)$$

or, there exists at least one $j \in \{1,2\}$ such that

$$J_j(u_1^*, u_2^*) < J_j(u_1, u_2)$$

3. (u_1^*, u_2^*) is called a Stackelberg solution with DM1 as leader and DM2 as follower if the leader's strategy satisfies

$$\sup_{u_2 \in R^2(u_1^*)} J_1(u_1^*, u_2) \leq \sup_{u_2 \in R^2(u_1)} J_1(u_1, u_2) \text{ for all } u_1 \in \underline{U}_1$$

where $R^2(u_1) := \{w \in \underline{U}_2 \mid J_2(u_1, w) \leq J_2(u_1, u_2) \text{ for all } u_2 \in \underline{U}_2\}$

is the rational reaction set of DM2.

The follower's strategy is obtained from

$$J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2) \text{ for all } u_2 \in \underline{U}_2.$$

c. The case of common objectives. Let $J := J_1 = J_2$

1. (u_1^*, u_2^*) is called a Team solution if

$$J(u_1^*, u_2^*) \leq J(u_1, u_2) \text{ for all } (u_1, u_2) \in \underline{U}_1 \times \underline{U}_2.$$

2. (u_1^*, u_2^*) is called a Person-by-Person Optimal solution if

$$J(u_1^*, u_2^*) \leq J(u_1, u_2^*) \qquad \text{for all } u_1 \in \underline{U}_1$$

$$J(u_1^*, u_2^*) \leq J(u_1^*, u_2) \qquad \text{for all } u_2 \in \underline{U}_2 .$$

□

A brief discussion of the solution concepts follows. The Nash equilibrium solution is applicable to a competitive situation. It displays the circumstance that a decision maker cannot benefit, if he unilaterally deviates from his equilibrium strategy, provided his opponent retains his own equilibrium strategy.

The Pareto (or: non-inferior) solution exhibits a cooperative situation. Any strategy different from a Pareto solution will yield higher costs for at least one player. Typically, Pareto solutions are not unique; we will show later how they can be parametrized.

The Team solution is applicable, if the decision makers have the same objectives, and is of special interest, when the decision makers do not share their information (see Marschak and Radner, 1972). For actual computation of the Team solution one must often resort to Person-by-Person-Optimal solutions (which is in fact the Nash solution for common objectives). A Person-by-Person Optimal solution is usually easier to compute than the Team Solution; the Team Solution implies the Person-by-Person Optimal Solution, but not vice versa (see Başar and Olsder, 1982, p. 187).

The Stackelberg solution applies to hierarchical or sequential situations and will not be discussed in this book (see De Zeeuw, 1984; Bagchi, 1984). Nor will we discuss the zero-sum case which is applicable in a persuit-evasion situation, or a situation when the players are antagonists.

3.3. The optimal control problem

In the previous section we have summarized the elements that constitute a stochastic dynamic game. A stochastic dynamic game is used as a mathematical model in the economic planning problem and serves the purpose of policy evaluation. Plans or decisions of an agent then follow from the solution of the optimal control problem that corresponds with the formulation of the stochastic dynamic game. One of the most important (but likely to be complicated) technical problems is how to solve this optimal control problem.

The statement of the optimal control problem requires the following preliminaries. The economic planning problem has been formalized into a mathematical model; this model is specified through the model set $\underline{M}$, the information patterns for on-line model data and the off-line model data of the decision makers. Furthermore, the decision makers are supposed to state cost functions J_1 and J_2 and strategy sets $\underline{U}_1$ and $\underline{U}_2$.

Then the optimal control problem consists of finding a solution (u_1^*, u_2^*) in $\underline{U}_1 \times \underline{U}_2$, subject to the completely specified stochastic dynamic game, given a particular solution concept.

If the optimal controls are unique and can be computed, and if the strategy sets and cost functions can be interpreted in terms of the economic planning problem, the theoretical findings can be exploited in real-world case studies.

However, most of the optimal control problems for stochastic dynamic games do not admit easily-implementable solutions. In the multi-agent case this is particularly so, if on- and off-line model data are non-shared. But even in the single-decision-maker case, difficulties arise when we depart from closed-loop or feedback strategies as in section 2.5 (see the classical example in Witsenhausen, 1968).

One way out of this dilemma is to find approximations to the optimal solution. If the optimal solution is known, but cannot be computed in any efficient way, then it may be approximated by a solution which is numerically more efficient. Also approximations at an earlier stage are feasible (for instance in the strategy sets, the information patterns, the model equations).

Another suggestion is to dismiss the notion of optimality. The fact that a rational economic agent maximizes his preferences is the key to, what is called, classical economic theory. However, in the control approach one may conceive the situation that a control strategy is suggested such that the controlled system has certain properties, in a qualitative sense mostly. This approach is common in the engineering literature, see Chen (1984), Callier and Desoer (1982), Kwakernaak and Sivan (1972). Some aspects may be meaningful for the economic planning problem and will be discussed in the next subsection.

3.4. Design criteria for control systems

Apart from the problem of optimal control, one of the most important issues in the engineering literature is the so-called design of control systems. We assume that a mathematical description of a physical or technical phenomenon is available (called a plant), and the problem is to find a control such that the plant behaves in a desired way. Criteria for desired plant behaviour are, among others, stability and sensibility of properties of the plant with respect to parameter changes and disturbances. In Kwakernaak and Sivan, 1972, chapter 2, a series of design objectives is proposed which cover most aspects of the desired behaviour.

Some of these design objectives seem appropriate to economic planning problems as well. Design criteria may be useful for the optimal control problem as well. It is possible that certain properties which we would like the controlled system to possess are not captured by the cost function. Therefore we propose a number of design and evaluation criteria which can be used as a checklist for both the optimal control and the design problem. We will distinguish four items, which may be of interest for the investigations of properties of the controlled system.

1. Qualitative behaviour of the controlled system
   a. Stability.
   b. Sensitivity of system properties to parameter changes.
   c. Sensitivity of system properties to uncertainty (e.g. expressed by the signal-to-noise ratio).
   d. How well does the system track the desired target and instrument paths? (especially the turning points).

2. Informational properties of the solution
   a. Does the decision maker use all available information?
   b. Is there a notion like the value of information in a multi-agent situation?
   c. Which is the most relevant information for a decision maker (i.e., the loss of information which would hurt him most when it is not available)?
   d. Does it pay to exchange or withhold information?

   Note that the items c. and d. depend on the cost function.

3. Numerical properties of the solution
   a. Can the algorithm obtained from the design or the optimal control solution be implemented and run on a middle-sized computer?
   b. Is the algorithm numerically stable?
   c. Is it advantageous or necessary to develop fast algorithms?

4. Usefulness of policy making
   a. Do the results lead to realistic recommendations for the decision maker?
   b. Does the practical evidence with the model lead to new views or improvements for the control methodology?

We want to emphasize that the main problem in this book is the solution of the optimal control problem for a particular stochastic dynamic game. In order to study the applicability of the method and to evaluate the actual application some criteria are required. A partial list is presented above. We will, however, not touch upon all of these issues.

CHAPTER FOUR

ANALYSIS OF THE CONTROL SYSTEM: INFORMATION AND TARGETS

4.1. Introduction

In this chapter we will analyse the specification of a stochastic dynamic game. In a specification as is meant here we can distinguish the class of mathematical models, the cost function and the solution concept. The former two concepts will be discussed in this chapter. The latter concept is not subject to discussion in this book, because we adopt the game-theoretical definitions (see definition 3.7). We only note that refinements of these concepts exist and are of importance, see Myerson (1977) or Van Damme (1984).

Concerning the class of mathematical models, we deal with ARX(p,q)-models and Gaussian system representations. For these classes, the notion of information specified by the information patterns is of major importance. In section 4.2 we will present a classification based on the sharing and non-sharing of information, and illustrate the relevance of the various cases for econometric modelling.

Concerning the cost function, we will give an interpretation in economic terms in section 4.3; simple, static models will be used to discuss the essential features of the cost function. We will also show the additional difficulties in analysis and interpretation which may be involved through the introduction of multiplicative noise. Conclusions will be summarized in section 4.4.

4.2. Specification of information in a stochastic dynamic game

4.2.1. Modelling conventions

Before we present the classification of a stochastic dynamic game based on the sharing and non-sharing of information, we will discuss the relation between on- and off-line model data. This relation is supposed to hold for the class of econometric models to be considered here (see also Chow, 1975; Intriligator, 1978).

Using the notation and definitions of chapter 3 we will adopt the following convention. Let $[0,t_0]$ be the sample period over which economic time series are given, represented in the form $\eta_{t_0} = \{y(s),\ u(s),\ s < t_0\}$. The values of the parameters are estimated from these economic data, subject to the choice of the model set $\underline{M}$. The result is a particular model $M(\theta)$. In this modelling convention the off-line model data θ are derived from the information specified by the information pattern for the on-line model data η_{t_0}.

We will assume that time t_0 is taken <u>fixed</u>. One can think of t_0 as the present. This assumption rules out recursive estimation procedures and adaptive control problems (see Kendrick, 1981), and constitutes a major simplification opposed to the case of a time-varying t_0.

In accordance with the assumption stated above, another major assumption must be stated explicitly: model $M(\theta)$ is supposed to describe the behaviour of the economy over the planning period $[t_0,t_f]$. Over this period optimal controls $u^*(t)$, $t \in [t_0,t_f]$ will be computed. When t_0-t_f is large (that is, we face a long-term planning horizon), it is unlikely that $M(\theta)$ renders a good description of the economy; structural changes in the economy may occur in that period. Therefore we will use the control approach only for medium- or short-term planning purposes.

Finally, we state that a class of admissible strategies must be specified for the planning period. We will assume that the information pattern for on-line model data determines this class. For example, when on-line model data for any $t \in [t_0,\ t_f]$ are of the form $\eta_t = \{y(s),\ u(s),\ s < t\}$, then we consider the partial-state observation case and take for $\underline{U}$ the class of closed-loop control laws (see section 2.5).

4.2.2. <u>A classification of stochastic dynamic games</u>

In definitions 3.2 and 3.3 we defined the information patterns for on- and off-line model data. If we apply the notion of sharing and non-sharing of information (see definition 3.5) to either of these concepts, we may distinguish four cases.

1. The global dynamics, shared information case.
   $\theta_1 = \theta_2$, $\eta_t^{(1)} = \eta_t^{(2)}$ for all $t \in T$.
2. The global dynamics, non-shared information case.
   $\theta_1 = \theta_2$, $\eta_t^{(1)} \neq \eta_t^{(2)}$ for at least one $t \in T$.
3. The local dynamics, non-shared information case.
   $\theta_1 \neq \theta_2$, $\eta_t^{(1)} \neq \eta_t^{(2)}$ for at least one $t \in T$.
4. The local dynamics, shared information case.
   $\theta_1 \neq \theta_2$, $\eta_t^{(1)} = \eta_t^{(2)}$ for all $t \in T$.

These four cases will be discussed separately below, and will form the basis for the formulation of several optimal control problems to be discussed in the following chapters.

Ad 1. <u>The global dynamics, shared information case</u>

We will discuss the case that the decision makers share both on-line model data $\eta_t = \{y(s), u(s), s \leqslant t\}$ and off-line model data θ. The following situation occuring frequently in econometric modelling is an example of this case.

Given the shared on-line model data, apply an estimation procedure to these data subject to the model set $\underline{M}$. A particular model $M(\theta)$ follows, to be used by the decision makers. In the case of ARX-models and Gaussian system representations, $M(\theta)$ can be represented as follows. Let $u = (u_1;u_2)$, $z = (z_1;z_2)$, such that u_i, z_i are instrument and target variables attributed to DMi, i = 1,2. The estimation procedure yields an ARX(p,q)-model which can be transformed to the Gaussian system representation

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u_1(t) + B_2u_2(t) + Mv(t) \\ y(t) &= Cx(t) + Du(t) + Nv(t) \\ \binom{z_1}{z_2}_t &= Hx(t) + (J_1 \;\; J_2)\binom{u_1}{u_2}_t \end{aligned} \tag{4.1}$$

The formulation of the optimal control problem for (4.1) can be completed by stating the class of admissible strategies, the cost functions and the solution concept. In chapter 5 we will discuss the following case: the class of admissible control laws is the class of closed-loop control laws in the partial-state observation case, the cost function is of the quadratic type for the regulator problem (section 2.5), the solution concept is the Nash or Pareto concept. These choices will admit analytic solutions suitable for implementation.

The assumption that economic agents share economic data, and that their behaviour is modelled by a common economic model is often found in econometric modelling. A few well-known examples will be given. We mention the project LINK by Klein (see Waelbroeck, 1976), the project COMET (see Barten et al., 1976) and the Interplay project (Plasmans, 1981). The latter model has been formulated as a dynamic game and used for policy evaluation (see De Zeeuw, 1984). Another example of a dynamic game application for monetary and fiscal authorities in the USA has been given in Pindyck (1977). The data for these models were supplied by international agencies (e.g. the National Accounts of the OECD and of Eurostat), or were obtained directly from the National Accounts of the countries under consideration.

Ad 2. <u>The global dynamics, non-shared information case</u>

An illustrative example of the global dynamics, non-shared information case can be derived from the system representation (3.1) in Example 3.4, with $\eta_t^{(i)} = \{y_i(s),\ u_i(s),\ s < t\}$, $i = 1,2$ and $\theta = \theta_1 = \theta_2 = \{A, B_1, B_2, C_1, C_2, D_1, D_2, M, N_1, N_2, V, m_0, \Sigma_0\}$.

Optimal control problems arising from this set-up have been analysed in the engineering literature. Particularly well-known is the so-called <u>decentralized control problem</u> in which is required that each decision maker uses his control based on past observations, such that the system (3.1) is stabilized (see Davison, 1976, and Davison and Özgüner, 1983 for an application to the regulation of traffic flows).

If we assume, however, that model parameters are estimated based on economic time series (as in section 4.2.1), then the global dynamics, non-shared information case does not seem to have much practical relevance for econometric modelling. Model (3.1) seems natural if we assume that a coordinator reveals the model parameters to all decision makers; or, if we assume that the model parameters follow directly from economic theory or from logical grounds, and not from statistical considerations.

One specific situation for which the global-dynamics, non-shared information case is suitable, will be mentioned. We consider an economic situation that will be modelled using the 1-step delayed observation sharing pattern (1sDOS), see definition 3.6.f. As a typical example, we discuss the construction of an econometric model for the European Community (EC). Data for such a model are obtained from the national accounts of the EC-members; these data are collected at a central agency (e.g. Eurostat, Luxemburg) and adjusted in order to obtain consistency. They are published with a time-lag of one sample period. When each country considers its last observation as non-shared information and all previous observations as shared, then the 1sDOS information pattern reflects this situation. A model of global dynamics type can be estimated based on the shared information.

The advantage of this set-up is that the optimal control problem for the 1sDOS-pattern can be solved analytically (see Başar, 1978; Kurtaran and Sivan, 1974, and Papavassilopoulos, 1981). Although the solutions proposed by the various authors do not completely agree and the resulting algorithm is rather complex, implementation is possible. It is of interest to compare the 1sDOS-solution with the standard LQG-solution; this will reveal the impact of private information (i.e. the most recent observation). Due to reasons of time and space we will not go further into this matter.

Ad 3. <u>The local dynamics, non-shared information case</u>

We will discuss the case in which both on- and off-line model data are non-shared. Let the information of DMi be represented by

$$\eta_t^{(i)} = \{y_i(s),\ u_i(s),\ s < t\} \text{ and } \theta_i,\ i = 1,2 \ .$$

According to the modelling conventions of section 4.2.1, we suppose that a decision maker estimates his model parameters based on economic time series available to him. Estimation based on non-shared on-line model data will result into non-shared off-line model data. Each decision maker is supposed to know the values of his parameters, but not the values of the parameters of the other decision maker (at least not all of them). In a more general set-up, one could even assume that the decision makers use different model sets; in that case the decision maker may not even know the model structure of the other decision maker, nor his solution concept, his class of admissible control laws and his cost function.

In the case of non-shared information, the results of the modelling efforts of the various decision makers are frequently represented as a collection of subsystems; in addition, some sort of interaction between the subsystems must be specified. In the literature this is known as a set of interconnected systems. The essential point is that it is not meaningful to compose the local systems into one overall system. This implies that standard approaches in game and control theory will fail. The subject of interconnected systems, however, has been treated in, what is called, Large-scale Systems Theory (see Jamshidi, 1983; Singh and Titli, 1978; Sandell e.a., 1978). Examples of practical applications are electric power-station networks (Davison and Tripathi, 1978) and water-resource systems (Haimes, 1977).

The local dynamics, non-shared information case is relevant for economic phenomena as well. In particular we mention duopoly and oligopoly situations. For example, two competitive firms operate on the same market. They keep their production plans and capacities secret, they try to maximize profits and to enlarge their market shares. Note that there is a piece of common knowledge, namely prices of products and, maybe, advertisement expenditures.

Finally, we remark that the notion of learning is essential in this approach. Whenever an economic agent receives some kind of information from the other agent's local system, then this information will be used to identify, in an adaptive manner, the model structure (parameters) of the other agent. In a highly simplified duopoly situation the practical fruitfulness of learning techniques has been indicated in Cyert and De Groot (1970). More recently, a fresh attempt to analyse the adaptive control problem for the multi-agent case has been proposed in Papavassilopoulos (1985). The topic of adaptive estimation and control will not be treated in this book (cf. section 4.2.1); therefore we will not go further into this matter.

Ad 4. The local dynamics, shared information case

It is assumed that the decision makers have shared on-line model data, but non-shared off-line model data. In the context of the proposed modelling conventions, the decision makers estimate different values for the parameters based on the same economic time series. This makes sense when the decision makers adhere to different economic theories. For example, one can think of an economic model based on Keynesian or monetarist theory. Such models have been called rival models in Rustem (1983). Recently, some additional information on rival models has been provided in Rustem (1985). On the same grounds as for the local dynamics, non-shared information case this topic will not be analysed here.

4.2.3. Conclusion

In this section we have considered four cases that have been obtained through a classification of information available to the decision makers. Depending on the particular problem at hand, one of these cases may be a suitable model for the economy. If we assume that economic models are constructed by estimation procedures using observed data, then the sharing (non-sharing) of on-line model data implies the sharing (non-sharing) of off-line model data. The two cases that arise seem to be of major importance for econometric modelling. The two other cases, however, could also be attributed a meaningful interpretation.

The types of problems met in the various cases differ greatly. A separate discussion of the cases will be presented, involving the different techniques to be used. Three cases will be considered in this book: the global dynamics, shared information case (chapter 5), the global dynamics, non-shared information case (chapter 6) and the local dynamics, non-shared information case (chapter 7).

4.3. The formulation of targets

4.3.1. Introduction

In section 2.5 we have formulated the LQG-problem for a cost function of the form

$$J(u) = \sum_{t=t_0}^{t_f} \{\|z(t) - \bar{z}(t)\|_Q^2 + \|u(t) - \bar{u}(t)\|_R^2\} \tag{4.2}$$

In this section we will analyse the form of the cost function. For convenience of exposition, we will assume that the vector u consists only of instrument variables. The following question will be addressed: can the desired values of the target variables ($\bar{z}(t)$, $t \in [t_0,t_f]$) be reached by a suitable choice of the instrument variables?

The problem whether desired values for target variables can be reached will be analysed in a setting of static policy models. This choice enables an analytic presentation, and facilitates interpretation. In addition we follow the historical development in quantitative economic planning (cf. Tinbergen, 1956).

Both deterministic and stochastic models will be considered. In the case of a deterministic model, we start by considering the question originally posed by Tinbergen. When the decision maker is assumed to use his instruments freely, can the desired target values be reached? Conditions for an answer in the affirmative sense will be given. In the case of an answer in the negative sense, the desired values for the target variables must be approximated; this can be done suitably by the introduction of a cost function. In this way the format of an optimal control problem is reached. The cost function displays the trade-off that the decision maker is supposed to make between the attainment of the desired values of the target variables and the use of the instrument variables.

In the case of a stochastic model, we will consider a simple model which contains both additive and multiplicative noise. As a prelude, the simpler case of only additive noise will be discussed in detail. The case of multiplicative noise is relevant due to the fact that parameters in the model may be obtained by an estimation method; the uncertainty in the parameters can then be taken into account. However, the corresponding control problem will turn out to be more complicated.

A brief outline of this section follows. In section 4.3.2 we treat various aspects of static, deterministic policy models. In section 4.3.3 we deal with static, stochastic policy models. The stochastic policy model will be introduced in subsection 4.3.3.1, the additive noise case will be treated in subsection 4.3.3.2, the multiplicative noise case in subsections 4.3.3.3 and 4.3.3.4. Conclusions and implications for dynamic models are given in subsection 4.3.3.5 and in section 4.4, respectively.

4.3.2. Static, deterministic policy models

4.3.2.1. Introduction

In this section we introduce the approach to quantitative economic planning as initiated by Tinbergen (1956).

The following relation between target and instrument variables will be considered. Consider (2.1) and (2.2) and ignore the uncontrollable exogenous variables. The instrument variables are denoted by u and the static, deterministic version to be derived from (2.1) and (2.2) is

$$y = Bu$$
$$z = Hy + Ju$$

After elimination of y we have

$$z = (HB + J)u \tag{4.3}$$

Expression (4.3) represents a linear relation between target variables z and instrument variables u. This argument is the motivation to study the so-called linear policy model

$$z = Pu \tag{4.4}$$

where $P: R^m \rightarrow R^r$ is a linear mapping from the action space $U \subseteq R^m$ to the target space $Z \subseteq R^r$. The decision maker is assumed to manipulate $z \in Z$ by means of $u \in U$.

Tinbergen addressed the following question: can the decision maker reach the desired value $\bar{z} \in Z$ by a suitable and cost-free use of his instrument variables? This is called the fixed target policy problem. If the desired value $\bar{z}$ cannot be reached, we are led to the flexible target policy problem: what action must be chosen to approximate $\bar{z}$?

4.3.2.2. The fixed target policy problem

In general, one may distinguish the problems of existence, uniqueness and design of a fixed target policy. In this section we will only deal with those aspects that are relevant for the formulation of targets.

Definition 4.1. The fixed target policy problem

Given the linear policy model

$$
\begin{aligned}
z = Pu, \; z &\in Z \subseteqq R^r \\
u &\in U \subseteqq R^m \\
P &: U \to Z
\end{aligned}
$$

and a fixed (desired) target value $\bar{z} \in Z$. Does a control $u \in U$ exist such that $\bar{z} = Pu$? If this is the case, is the control value unique, and how can it be computed?

Definition 4.2

The fixed target policy problem is called

a. locally solvable at $z = \bar{z}$ for some $\bar{z} \in Z$ if a $u \in U$ exists such that $\bar{z} = Pu$.

b. globally solvable if for all $z \in Z$ a $u \in U$ exists such that $z = Pu$.

From here on we will assume that we deal with the full spaces $Z = R^r$ and $U = R^m$.

Proposition 4.3

a. The fixed target policy problem is
   1. locally solvable at $z = \bar{z}$ iff $\mathrm{rank}[P,\bar{z}] = \mathrm{rank}[P]$
   2. globally solvable iff $\mathrm{rank}[P] = r$.
b. The fixed target policy problem has at the most one solution for all $z \in Z$ iff $\mathrm{rank}[P] = m$.
c. The fixed target policy problem has exactly one solution for all $z \in Z$ iff $m = r = \mathrm{rank}[P]$.

Proof. This is a fundamental result in the theory of linear algebraic equations, see Strang, 1980, ch. 2.

□

From part a of the proposition we observe that global solutions may exist whenever $r \leqslant m$, i.e. the number of target variables falls below the number of instrument variables. The special case $m = r = \mathrm{rank}[P]$ was the starting point of Tinbergen's analysis. He assumed an equal number of target and instrument variables and a nonsingular policy matrix P. The design problem is then solved at once: $u = P^{-1}z$.

A complete characterization of the solution as well as of the design problem for all cases ($m < r$, $m = r$, $m > r$, P having full or deficient rank) can be obtained using generalized inverses. Instead of going into these technical matters we will discuss a procedure that is valid when global solutions do not exist.

4.3.2.3. The flexible target policy problem

It has been observed by several authors that the case treated by Tinbergen was rather restrictive. Therefore Theil proposed the so-called flexible targets (Theil, 1964). The instrument variables can be used to approximate the desired value $\bar{z}$ of the target variables. A preference function, attributed to the economic agent, states in what sense this approximation must be made. The following two examples will illustrate this approach and clarify how the problem can be formulated as a constrained minimization problem.

Example 4.4

Consider the linear policy model $z = Pu$, with $r = 2$, $m = 1$.
Let $z = (z_1;z_2)$. The relation $z = Pu$ can be represented by a point in the (z_1,z_2)-plane, for a fixed value of u. The locus of points which arises by varying u over U form a straight line; in figure 4.1 it is designated by Pu.

The desired target value $\bar{z} = (\bar{z}_1;\bar{z}_2)$ will generally not be located on this line. In this case the economic agent must decide to approximate $\bar{z}$ by a point on Pu (see figure 4.1).

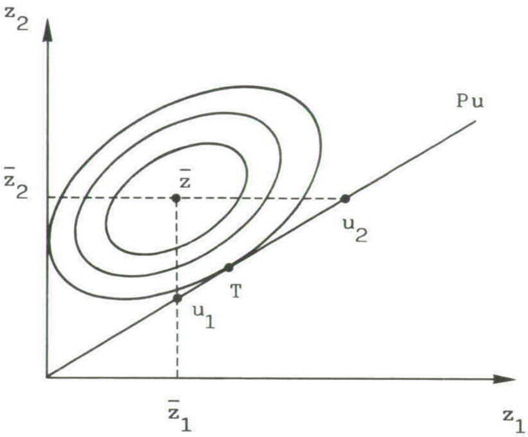


Figure 4.1. The linear policy model $z = Pu$ ($r = 2$, $m = 1$).

Either $\bar{z}_1$ can be reached by u_1, or $\bar{z}_2$ can be reached by u_2, but not both. The problem of which u to choose may be solved in the following way. It is assumed that the decision maker can rank the possible combinations (z_1,z_2) into a quadratic preference function $(z-\bar{z})^T Q(z-\bar{z})$, $Q = Q^T \in R^{2\times 2}$. Contours or indifference curves of such preference functions are shown by ellipses in figure 4.1, representing increasing levels of utility. The tangent point T is the point on Pu with the highest utility.

□

Note that in the example above, J does not depend on u which is in contrast to (4.2). This will be remedied in the next example.

Example 4.5

We will consider an economic example, due to Holt (1962). The policy model is supposed to be

$$sY - G = I \tag{4.5}$$

where Y is GNP (Gross National Product, a target variable), G is government expenditures (an instrument variable), I is investments (an uncontrollable exogenous variable), and s is a multiplier ($s > 0$). Let $\bar{Y}$ be the desired level of GNP and $\bar{G}$ the desired value for governmental expenditures. The choice $G = \bar{G}$ reflects the fact that the government faces other constraints than imposed by the model (4.5), e.g. due to political and/or social considerations. The pair $(\bar{Y},\bar{G})$ is called non-feasible if it does not satisfy (4.5).

Note that (4.5), with desired values $(\bar{Y},\bar{G})$, differs from the previous model in two ways. First, it is of the form $z = Pu + d$, with d a known uncontrollable exogenous variable. Secondly, both z and u have a desired value. In our example, the desired value $\bar{Y}$ can be reached by $\tilde{G}$; the desired $\bar{G}$ will yield a GNP of $\tilde{Y}$, see figure 4.2.

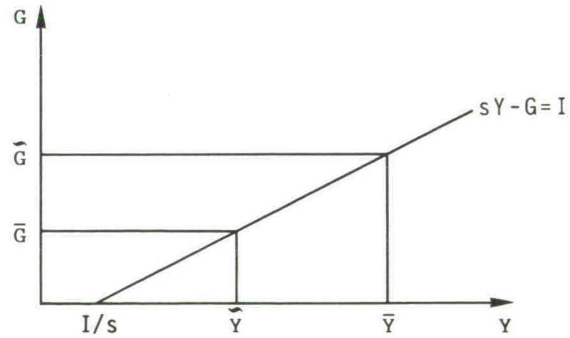


Figure 4.2. The linear policy model $z = Pu + d$ ($r = 1$, $m = 1$).

Let us proceed analogously as in Example 4.4 and state a preference function for the government of the following type.

$$J(G) = (Y-\bar{Y})^2 + \theta(G-\bar{G})^2$$

where $\theta \geq 0$ is a weighting parameter. Minimization of $J(G)$ with respect to G subject to $sY - G = I$ yields the linear decision rule for the optimal G^*.

$$G^* - \bar{G} = \frac{s}{1 + \theta s^2} (\bar{Y} - \tilde{Y}) \tag{4.6}$$

$G^* - \bar{G}$ reflects the deviation of the optimal instrument setting from the desired value for the instrument variable; it is linear in $(\bar{Y}-\tilde{Y})$ which reflects the inconsistency in the values $\bar{Y}$ and $\tilde{Y}$. These values are both desired by the decision maker; the former by definition, the latter in order to "obtain" $\bar{G}$.

Note that θ reflects the trade-off between the attainment of $\bar{G}$ and the attainment of $\bar{Y}$. If $\theta \to \infty$ then $G^* \to \bar{G}$, and if $\theta \to 0$ then $G^* \to \tilde{G}$.

□

From Example 4.5 we observe that instrument variables can also have a role as target variables. Formally, we may as well treat them as target variables. Hence rewrite (4.5) as the augmented policy model

$$Y = (I+G)/s$$
$$X = G$$

Using the terminology $z = Pu + d$ for a linear policy model with an uncontrollable exogenous input d, we have

$$z = (Y;X), \quad \bar{z} = (\bar{Y};\bar{G})$$
$$u = G$$
$$P = (1/s;1), \quad d = (I/s;0)$$

Using the result of Proposition 4.3 we conclude that no globally solvable fixed policies will exist. This conclusion also holds for a policy model with an arbitrary number of target and instrument variables, as soon as a desired value is specified for any of the instrument variables. Hence we propose to use the approach of flexible targets; from Example 4.5 we observe that the mathematical problem is in fact a constrained minimization problem. The general case for models with desired instrument values is considered below (cf. Theil, 1964, ch. 2).

Proposition 4.6

Given the linear policy model $z = Pu$, $z \in Z = R^r$, $u \in U = R^m$, with desired values $(\bar{z},\bar{u})$ for target and instrument variables, and cost function

$$J(u) = (z-\bar{z})^T Q(z-\bar{z}) + (u-\bar{u})^T R(u-\bar{u}) \tag{4.7}$$

where $Q \in R^{r\times r}$ and $R \in R^{m\times m}$ are symmetric weighting matrices.

If $(R + P^T QP)$ is nonsingular, then (4.7) is minimized subject to the constraint $z = Pu$, by

$$u^* = (R + P^T QP)^{-1}(P^T Q\bar{z} + R\bar{u}) \tag{4.8}$$

Proof. For the cost function (4.7) and the constraint (4.4) define the Lagrangean:

$$L(z,u,\lambda) := J(u) + \lambda^T(z-Pu)$$

where $\lambda \in R^r$ is the Lagrange multiplier.
The first-order conditions are

$$\frac{\partial L}{\partial z} = 2(z-\bar{z})^T Q + \lambda^T = 0$$

$$\frac{\partial L}{\partial u} = 2(u-\bar{u})^T R - \lambda^T P = 0$$

$$\frac{\partial L}{\partial \lambda} = z - Pu = 0$$

Reorganizing yields

$$(R + P^T QP)u = P^T Q\bar{z} + R\bar{u} \tag{4.9}$$

from which (4.8) follows, if $(R + P^T QP)$ is nonsingular.

□

Remarks. Nonsingularity of $(R + P^TQP)$ is implied by the well-known conditions $Q \geq 0$ and $R > 0$. In that case $(R + P^TQP) > 0$.
If we define $\tilde{z}$ by $\tilde{z} = P\bar{u}$, (4.8) can be rewritten as the linear decision rule, cf. (4.6)

$$u^* - \bar{u} = (R + P^TQP)^{-1}P^TQ(\bar{z}-\tilde{z}) .$$

Note that (4.9) is of the same algebraic form as $z = Pu$, and may be considered as a fixed target policy model. The desired fixed target is now $P^TQ\bar{z} + R\bar{u}$, the policy mapping is $(R + P^TQP)$, and the design problem is solved by (4.8). Hence there exists a formal analogy between the solutions of the fixed and the flexible target policy problem.

4.3.2.4. Conclusions for the static, deterministic policy model

In a static, deterministic policy model the desired values of the target variables cannot be reached, in general, by a suitable choice of instrument variables. Hence, the desired target values must be approximated, which implies a trade-off between the attainment of the desired target values and the use of the instrument variables. This trade-off is made explicit by the introduction of a cost function (preference function). In fact we have arrived at a constrained optimization problem. Analytic results for this type of decision problems are at hand, if a quadratic cost function is chosen together with a linear decision model as a constraint. The optimal solution is easily obtained by invoking Lagrangean theory for constrained optimization problems and can be presented in the attractive format of linear decision rules.

The desired values of the target variables cannot be reached if we make the following model assumption. In the formulation of the cost function desired values are stated for all target variables and for all instrument variables. An economic motivation for this situation follows: the decision maker faces political, social, financial constraints in the use of his instruments. Beyond their use of manipulating the system's state, the instrument variables have to be treated as target variables as well. From the fixed target policy problem it is known that the desired target values usually cannot be reached in this case.

4.3.3. Static, stochastic policy models

4.3.3.1. Introduction

In this section we will introduce uncertainty into the policy model. The results of this section are only for explanatory reasons and not for developing a general theory. Therefore the simplest forms for the model equation and the cost function have been chosen. An economic interpretation in terms of a related monetary model has been given by Brainard (1967).

The stochastic policy model is given by

$$z = au + v \tag{4.10}$$

and the cost function is given by

$$J(u) = q(z-\bar{z})^2 \tag{4.11}$$

where $z : \Omega \to R$ is the target variable, $\bar{z}$ its desired value, $a : \Omega \to R$ is a stochastic multiplier with mean μ_a and finite variance σ_a^2 (the notation $a \in L^2(\mu_a,\sigma_a^2)$ will be used), $u \in U = R$ is the instrument variable, and $v : \Omega \to R$ with $v \in L^2(\mu_v,\sigma_v^2)$ is the random disturbance. Let ρ be the correlation coefficient between a and v and assume that $q > 0$, $\mu_a > 0$, $\bar{z}-\mu_v > 0$. The decision maker is supposed to know q, $\bar{z}$, μ_a, σ_a^2, μ_v, σ_v^2 and ρ.

In this simple set-up all variables are scalar, and the cost function does not depend on u explicitly. Although a more general model does not prohibit mathematical results, it does obscure the graphical presentation and corresponding insight that will be presented below.

We will analyse the problem

$$\underset{u \in U}{\text{minimize}}\ E[J(u)] \text{ subject to } z = au + v \tag{4.12}$$

A simpler case will be treated first. Assume that $\sigma_a^2 = 0$, and refer to this case as the <u>additive noise case</u> (because a can now be considered as a deterministic coefficient). From (4.10) it can be seen that the control u determines only the mean of the distribution of z. Hence the problem is similar to the fixed target policy problem of section 4.3.2.2: how to use u such that the expected value of z is $\bar{z}$? Secondly, we will treat the general case: the <u>case of multiplicative noise</u> ($\sigma_a^2 > 0$). The instrument variable u will not only affect the mean but also the form of the distribution of z. This dual role of u must be emphasized: the decision maker will use his instrument variables such that the expected value of z tends towards $\bar{z}$, and such that the uncertainty about the value that z will assume, measured by the variance σ_z^2, will be reduced. We will analyse the trade-off between E[z] and σ_z^2 in some detail. Note that, due to the form of (4.10) and (4.11), we are in the case of one target variable and one instrument variable. Hence we may compare the outcomes of the stochastic policy model with the fixed target policy problem of section 4.3.2.2.

4.3.3.2. <u>The additive noise case:</u> $\sigma_a^2 = 0$

For the stochastic policy model with $\sigma_a^2 = 0$ a <u>certainty equivalence</u> result may be derived. We will first define certainty equivalence in a general setting, in which z = f(u,v) is the relation between the target variables z, the control u and the noise v, and W(u) is a general cost function.

<u>Definition 4.7</u>

If the minimizing controls of the two minimization problems

1. $\min_{u \in U} E[W(u)]$ subject to z = f(u,v)

2. $\min_{u \in U} W(u)$ subject to z = f(u,E[v])

are equal, then the minimizing control is called static certainty equivalent.

□

Proposition 4.8

Consider the minimization problem (4.12) with $\sigma_a^2 = 0$ and $a = \mu_a$ known. Then

a. The optimal control u_{CE}^* is given by $u_{CE}^* = (\bar{z}-\mu_v)/a$.
b. u_{CE}^* is static certainty equivalent.
c. Under u_{CE}^* the mathematical expectation of the target variable equals its desired value.

Proof.

a. In order to solve (4.12) we compute $E[J(u)]$.

$$E[J(u)] = q[a^2u^2 + 2a\mu_v u - 2a\bar{z}u + E[v^2] + \bar{z}^2 - 2\mu_v\bar{z}]$$

Minimization with respect to u yields

$$\frac{dE[J(u)]}{du} = 2q[a^2u + a\mu_v - a\bar{z}] = 0 \ .$$

Hence $u_{CE}^* = (\bar{z}-\mu_v)/a$.

b. The minimization of $q(z-\bar{z})^2$ subject to $z = au + \mu_v$ yields:

$$u^* = (\bar{z}-\mu_v)/a \ .$$

c. $E[z] = au + \mu_v$; hence for $u = u_{CE}^*$ we have

$$E[z]\Big|_{u=u_{CE}^*} = a(\bar{z}-\mu_v)/a + \mu_v = \bar{z}$$

□

Remark

Note that the certainty equivalence result is powerful. It states that a stochastic minimization problem can be solved by replacing all random variables by their expectations and then by solving a deterministic minimization problem.

The original formulations of the certainty equivalence result, similarly as presented here, can be found in Simon (1956), Theil (1957) and Theil, 1964, section 2.2. Their work has been followed by many attempts to prove a certainty equivalance result for wider classes of control problems. See Bar-Shalom and Tse (1974) for definitions and some results (see also Appendix 5E).

4.3.3.3. The multiplicative noise case: $\sigma_a^2 > 0$

We shall solve problem (4.12), i.e. the minimization of $E[J(u)]$ subject to $z = au + v$, in the case of a stochastic multiplier a. Recall from subsection 4.3.3.1 that we have assumed that $\mu_a > 0$, $\bar{z} - \mu_v > 0$, $q > 0$.

Proposition 4.9

Consider the minimization problem (4.12), with $a \in L^2(\mu_a, \sigma_a^2)$.
The optimal control u^* is given by

$$u^* = \frac{(\bar{z}-\mu_v)\mu_a - \rho\sigma_a\sigma_v}{\sigma_a^2 + \mu_a^2} \tag{4.13}$$

Proof.

$$E[J(u)] = q[(\sigma_a^2 + \mu_a^2)u^2 + 2u(\rho\sigma_a\sigma_v + \mu_a\mu_v)$$

$$-2u\bar{z}\mu_a + \sigma_v^2 + \mu_v^2 + \bar{z}^2 - 2\bar{z}\mu_v]$$

Hence,

$$\frac{dE[J(u)]}{du} = 2q[(\sigma_a^2 + \mu_a^2)u + \rho\sigma_a\sigma_v + \mu_a\mu_v - \bar{z}\mu_a] = 0$$

and the result follows.

□

The result can readily be interpreted if we consider the special case $\rho = 0$, i.e. the stochastic multiplier a and the noise term v are uncorrelated. Then we have from (4.13)

$$u^* = \frac{(\bar{z}-\mu_v)}{\mu_a + \frac{\sigma_a^2}{\mu_a}} < u^*_{CE} = \frac{\bar{z} - \mu_v}{\mu_a} ,$$

since we have assumed that $\mu_a > 0$. We conclude that the optimal control is not static certainty equivalent in this case. In fact, due to the uncertainty caused by a, the optimal control is more cautious compared to u^*_{CE}.

Under the control u^* the desired target $\bar{z}$ will not be reached in expectation, since from (4.10) and (4.13) for $\rho = 0$:

$$E[z]\Big|_{u=u^*} = \frac{\bar{z} - \mu_v}{1 + \sigma_a^2/\mu_a^2} + \mu_v < \frac{\bar{z} - \mu_v}{1} + \mu_v = \bar{z}$$

The case $\rho \neq 0$ can be treated similarly. If $\rho > 0$, the value of the optimal control is even more reduced compared to the case $\rho = 0$ (see (4.13)). This is due to the fact that high values of v correlate with high values of a, and this makes the decision maker even more cautious. If $\rho < 0$, the noise influences are reduced to some extent, and u^* tends to the certainty equivalent value u^*_{CE}.

The results in the multiplicative noise case display the following trade-off: the control action serves to reach the target (measured by $E[z]$) and to reduce the uncertainty about the value to be assumed by the target variable (measured by σ_z^2). This relationship will be explored in some detail in the next section.

4.3.3.4. The trade-off between $E[z]$ and σ_z^2

The relationship between $E[z]$ and σ_z^2 will be illustrated by means of graphs. The analytic relation between $E[z]$ and σ_z^2 follows from (4.10) and (4.11), if we eliminate u.

From (4.10): $E[z] = \mu_a u + \mu_v$ and (4.14a)

$$\sigma_z^2 = \sigma_a^2 u^2 + \sigma_v^2 + 2\rho\, \sigma_a \sigma_v u \tag{4.14b}$$

From (4.11): $\frac{E[J(u)]}{q} = \sigma_z^2 + (E[z] - \bar{z})^2$ (4.15)

Elimination of u from (4.14a) and (4.14b) yields

$$E[z] = \mu_v + \frac{\mu_a}{\sigma_a} [-\rho\sigma_v \pm \sqrt{\sigma_z^2 - (1-\rho^2)\sigma_v^2}] \tag{4.16}$$

(4.15) and (4.16) can be interpreted as relations between E[z] and σ_z. Indeed, for a fixed value of E[J(u)]/q, (4.15) represents a circle with centre $(0,\bar{z})$ in the $(E[z],\sigma_z)$-plane. Contours or indifference curves appear up by varying the costs in (4.15). (4.16) represents a hyperbola with centre $(0,\mu_v - \rho \frac{\mu_a\sigma_v}{\sigma_a})$ and vertex

$$(\sqrt{1-\rho^2}\ \sigma_v,\ \mu_v - \rho \frac{\mu_a\sigma_v}{\sigma_a}) \tag{4.17}$$

The slopes of the asymptotes of the hyperbola are given by $\pm\ \mu_a/\sigma_a$, hence independent of ρ.

The case $\rho = 0$ will be treated first.

A. <u>The case $\rho = 0$</u>

In the case of $\rho = 0$, (4.16) becomes

$$E[z] = \mu_v \pm \frac{\mu_a}{\sigma_a} \sqrt{\sigma_z^2 - \sigma_v^2} \tag{4.18}$$

The circles and the hyperbola, to be derived from (4.15) and (4.18) respectively, will be depicted in the $(E[z],\sigma_z)$-plane.

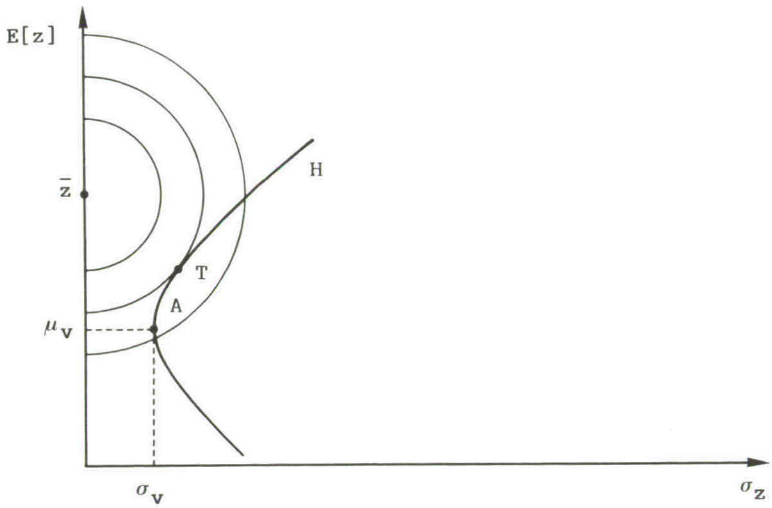


Figure 4.3. The trade-off between E[z] and σ_z for $\rho = 0$.

Figure 4.3 can be interpreted in the following way. (4.18) now represents hyperbola H with vertex $A(\sigma_v, \mu_v)$ and slopes of the asymptotes $\pm \mu_a/\sigma_a$. (4.15) is represented by circles with centre $(0, \bar{z})$. By varying the control u the hyperbola H will be traversed. The value $u = 0$ corresponds with the point A; for positive (negative) u the upper (lower) part of the hyperbola will be traversed. Any point on H in figure 4.3 has three meanings: first, it represents a certain control action; secondly, its coordinates are the expected value and the uncertainty of the target variables; thirdly, it yields costs, the value of which can be found by intersection with the indifference curves.

It is obvious that the tangent point T designates the optimal u^*. From the results of section 4.3.2 we know that the decision maker will not expect to reach $\bar{z}$. This is apparent from figure 4.3. We also note that the uncertainty by which the target variables will be reached is larger than σ_v which is the uncertainty incurred for $u = 0$.

B. <u>The case $\rho \neq 0$</u>

Consider figure 4.4.

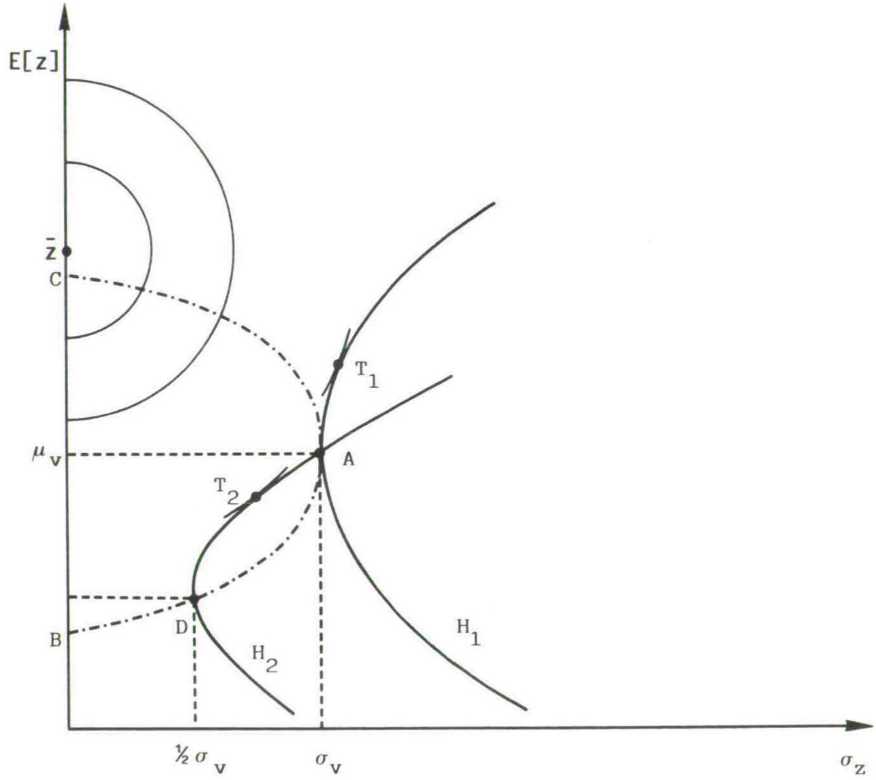


Figure 4.4. The trade-off between E[z] and σ_z, for $\rho = 0$ and $\rho = \frac{1}{2}\sqrt{3}$.

From (4.17) we observe that the vertex of the hyperbola depends on ρ. The locus of vertices, when ρ varies over $[-1,+1]$ is an ellipse with centre $(0,\mu_v)$.
A, B and C are points on the ellipse, such that

$$A = A(\sigma_v,\mu_v) \text{ for } \rho = 0 \;,$$

$$B = B(0,\mu_v - \sigma_v\mu_a/\sigma_a) \text{ for } \rho = +1 \;,$$

$$C = C(0,\mu_v + \sigma_v\mu_a/\sigma_a) \text{ for } \rho = -1 \;.$$

For two values of ρ the hyperbola represented by (4.16) is depicted.

For $\rho = 0$, we have hyperbola H_1 with vertex $A = A(\sigma_v, \mu_v)$ and tangent point T_1 for $u = u_1^*$.
This is identical to figure 4.3.
For $\rho = \frac{1}{2}\sqrt{3}$ we have hyperbola H_2 with vertex
$D = D(\frac{1}{2}\sigma_v, \mu_v - \frac{1}{2}\sqrt{3}\,\mu_a\sigma_v/\sigma_a)$ and tangent point T_2 for $u_2 = u_2^*$.
Note that H_2 (in fact: every hyperbola) passes through A which corresponds with $u = 0$.

Let us consider the cases $\rho = 0$ and $\rho = \frac{1}{2}\sqrt{3}$ depicted in fig. 4.4, and compare the actions $u = 0$, $u = u_1^*$ and $u = u_2^*$.
For $u = 0$, the result for both values of ρ is $A(\sigma_v, \mu_v)$, or

$$E[z] = \mu_v$$

$$\sigma_z = \sigma_v \ .$$

Let $\rho = 0$, then under $u = u_1^*$ we have (for tangent point T_1)

$$E[z]\Big|_{u_1=u_1^*} > \mu_v$$

$$\sigma_z\Big|_{u=u_1^*} > \sigma_v$$

The action u_1^* steers the target variable towards $\bar{z}$ at the expense of an increased uncertainty about the value that the target variable will assume.
Let $\rho = \frac{1}{2}\sqrt{3}$, then under $u = u_2^*$, we have (for tangent point T_2)

$$E[z]\Big|_{u_2=u_2^*} < \mu_v$$

$$\sigma_z\Big|_{u_2=u_2^*} < \sigma_v$$

Now the action u_2^* steers the target variable <u>away</u> from $\bar{z}$, in order to reduce the uncertainty about the target variable.

We conclude that this latter possibility departs sharply from the outcome in the comparable certainty or certainty equivalence situation.

4.3.3.5. Conclusions for the static, stochastic policy model

A stochastic policy model has been analysed with respect to the question whether the decision maker expects to reach the desired values of the target variables.

In the case of additive noise, we have established the static certainty equivalence result. This result implies that the optimal control can be obtained by solving a corresponding deterministic problem. In the case of one target variable and one instrument variable, the decision maker will expect to reach the desired value of the target variable. This result parallels the result for the static deterministic policy model of section 4.3.2. In the case of an arbitrary number of target and instrument variables, a trade-off between the attainment of the target variables and the use of the instrument variables must be made as has been done in the deterministic policy model. The incorporation of additive noise does not change the value of the optimal decision; only the costs incurred rise.

In the case of additive and multiplicative noise, i.e. a policy model with stochastic parameters, an additional trade-off must be made beyond the one between the target variables and the instrument variables. On the one hand, the target variable must be steered towards its desired value; on the other hand, the uncertainty about the value to be assumed by the target variable must be reduced. This is called the dual role of control. Examples can be conceived such that the decision maker primarily focusses on the uncertainty of the target variable, rather than steering towards the desired target value. This implies that the appealing intuitive notion that the economy is propelled towards the desired values of the target variables may be frustrated.

4.4. Summary and conclusions

The specification of information and the formulation of targets in a control system have been discussed in section 4.2 and 4.3, respectively.

Concerning the specification of information, we have proposed a classification based on sharing and non-sharing of the on-line and the off-line model data. Three cases shall be investigated.

1. The global dynamics, shared information case

   This seems to be the major application in econometric modelling: all decision makers share the on-line and the off-line model data. Applicability and implementation for real-world economic systems is feasible; a well-known example is the case of a linear model, quadratic cost functions and Gaussian disturbances. In the LQG-setting tractable control algorithms will be developed for the Nash and the Pareto concepts in chapter 5.

2. The global dynamics, non-shared information case

   The corresponding control problem will be stated and analysed; in particular we will deal with the special case in which restrictions have been imposed on the strategy sets. Although this restriction leads to a simplification of the calculation of the (sub)optimal controls, compared to the general unrestricted case, still difficult numerical problems will arise. We refer to this type of control problems as the Restricted Control Problem which will be studied in chapter 6.

3. The local dynamics, non-shared information case

   Each decision maker is assumed to construct his own model based on (partially) private information. An explicit account for the interaction must be given. We conceive this model as being composed of a set of interconnected systems, a topic discussed in Large-Scale System Theory. For this subject no full theory is available; however, for some isolated problems ad-hoc results exist. Some of these problems will be discussed in chapter 7.

We have proposed a classification from which three different types of control problems will be considered. In section 3.2 we have noticed that the main ingredients for a multi-decision-maker control problem were the solution concept, the information patterns and the cost functions. The first two concepts have now been dealt with extensively, in sections 3.2.2 and 4.2 respectively. The form of the cost function has been analysed in section 4.3. Note that the results from that section were derived for a static policy model. The implications for the dynamic case require further analysis. Below we will draw conclusions as to how to formulate targets which seem valid for both the static and the dynamic case.

The analysis of section 4.3 focussed on the role of $(\bar{z},\bar{u})$. We have shown that, in general, the desired target value $\bar{z}$ (or, in a dynamic setting, the desired target path) cannot always be reached. This problem is resolved by the formulation of an optimal control problem: a cost (preference) function is attributed to the decision maker, who must perform a trade-off between the attainment of $\bar{z}$ and $\bar{u}$. If we restrict our attention to the additive noise case, the instrument variables can be handled in order to steer the economy towards $\bar{z}$. For stochastic parameter models, however, the uncertainty by which $\bar{z}$ will be reached must be taken into account. In order to avoid this additional complexity we have only treated the additive noise case.

Let us now consider the terms of the cost function, see (4.2). The interpretation of $(z(t) - \bar{z}(t))^T Q(z(t) - \bar{z}(t))$ is obvious: deviations from $\bar{z}(t)$ are penalized, and the decision maker attempts to steer the economy towards $\bar{z}(t)$.

For the interpretation of $(u(t) - \bar{u}(t))^T R(u(t) - \bar{u}(t))$ a number of arguments are available in the literature.

First, in most practical applications the use of the instruments is limited (manifested by restrictions on the action set U). Rather than the introduction of inequalities for u, we introduce the term $(u(t) - \bar{u}(t))^T R(u(t) - \bar{u}(t))$ into the cost function. Hence, as for the target variables deviations from the desired instrument values $(\bar{u}(t), t \in [t_0,t_f])$ are penalized in the cost function.

Secondly, this extra term is interpreted as the instrument costs. The use of the instruments may not be free of costs. In addition, some instruments may be more easily handled than other ones. In the model this can be reflected by the differentiation of the elements of the weighting matrix R.

Thirdly, if we set R = 0 the instrument variables may exhibit violent variations. Under certain conditions, however, the optimal control problem for R = 0 is still tractable, see Chow (1975). In a dynamic control problem volatile behaviour of u(t) may be countered by replacing $(u(t) - \bar{u}(t))^T R(u(t) - \bar{u}(t))$ by $(u(t) - u(t-1))^T R(u(t) - u(t-1))$; this term prohibits violent variations in u and guarantees a smooth path for u without the necessity to state $(\bar{u}(t), t \in [t_0, t_f])$. Note that, on the other hand, a decision maker may want to impose shocks on the economy; in that case such a formulation of the cost function would be undesirable.

All these arguments have the common perspective that the control method must lead to practical results in application studies. From the variety of alternatives to achieve this aim we have chosen a quadratic cost function and specification of the desired path $(\bar{z}(t), \bar{u}(t), t \in [t_0, t_f])$. In combination with a linear model and suitably chosen information patterns the optimal control problem will lead to the attractive and practically useful linear decision rule. This approach has proved its feasibility in many practical applications.

CHAPTER FIVE

THE GLOBAL DYNAMICS, SHARED INFORMATION CASE

5.1. Introduction

In this chapter we discuss the situation that the decision makers act according to the same model, and share their on-line model data. It will turn out that we can state and solve the optimal control problem for this case. And since it is believed that this is one of the most interesting cases for the econometric practice, we will present a broader perspective than merely the solution for the optimal control problem and discuss two additional topics.

First, we shall review the transformation of an econometric model to state-space form. The ARX(p,q)-model will be taken as a result from the econometric estimation methodology (cf. section 2.4). The Gaussian system representation provides the most suitable form for the formulation of the optimal control problem. We will explore the consequences for the formulation of the control problem, if we start from an ARX-model followed by a transformation to state-space form, and if we start from a Gaussian system representation directly. The main result is that formally both approaches are equal, but that their interpretations differ.

Secondly, the properties of the resulting state-space form will be analysed from a computational viewpoint. The central concept is the notion of state. We will explore properties that reveal the relation between the input variables and the state, and between the state and the target variables. Except for an insight into the quantitative relation between input variables and target variables the state and, in particular, its dimension will determine the computational performance of the optimal control algorithm. Therefore it is desirable to obtain state-space models with a low dimension. This can be achieved by a suitably chosen transformation or by model reduction techniques. An analysis of the relevant properties of the system representation might reveal whether model reduction is a viable suggestion. For large macroeconometric models this seems indeed the case; the essential dynamics are usually described by a small subset of the model equations.

Let us return to the optimal control problem. Two solution concepts will be considered: the Nash equilibrium concept and the Pareto concept (see definition 3.7). For these two concepts optimal control solutions will be presented that are slight modifications of known results in the literature (Başar and Olsder, 1983). The modifications are due to the fact that the cost functions have a form slightly different from usual cost functions. The solution is based upon stochastic dynamic programming as formulated by Striebel (1975). In addition, results from the Kalman filter are required.

The organisation of this chapter is as follows. In section 5.2 we present the transformation of an ARX(p,q)-model to a Gaussian system representation; in section 5.3 the dynamic properties of such a representation are explored. In section 5.4 optimal control algorithms are presented for the Nash and Pareto concepts. The single-decision-maker LQG-problem is presented in full detail in the appendices. In Appendix 5A we discuss the abstract control system and optimality conditions, due to Striebel, 1975. In Appendix 5B the Kalman filter equations are given, and in Appendix 5C the complete solution to the LQG-problem is given. In Appendix 5D we state the (dynamic) certainty equivalence result for the LQG-problem. In Appendix 5E the derivation of the optimal Nash solution is given.

5.2. ARX-models and Gaussian system representations

We will show how to formulate the optimal control problem for a stochastic dynamic game in the global dynamics, shared information case. The on-line model data are $\{y(s), u(s), s \leq t\}$, where y are the (shared) observations of DM_1 and DM_2 and $u = (u_1; u_2)$ with u_i the input of DM_i. y and u may also be conceived as endogenous and exogenous variables, respectively.

Two ways lead to the optimal control problem. First, the econometric model is represented as an ARX(p,q)-model, whose parameters are estimated from the data $\{y(s), u(s), s \leq t\}$. The ARX(p,q)-model will be transformed to a Gaussian system representation. The specification of the target variables, of the class of admissible strategies, of the cost functions of and the solution concept completes the formulation of dynamic game. Secondly, the Gaussian system representation can be introduced at the outset. Its parameters can be estimated from data specified by the information pattern for on-line model data. Then we can proceed as above.

These two approaches will be confronted. They are different because they start from two models based on different assumptions. However, we will show that we end up with one formulation of the optimal control problem.

The programme in this section is as follows. In section 5.2.1 we start from an ARX(p,q)-model and show, in a series of propositions, how it can be converted to a Gaussian system representation. In section 5.2.2 we will confront the optimal control problem that is obtained when we start from an ARX(p,q)-model, proceeding via the state-space form, to the one that is obtained when we start directly from the state-space form.

5.2.1. The transformation of ARX(p,q)-models

In this section the ARX(p,q)-model given by (2.1) will be transformed to state-space form. Because the resulting state-space form will be used for computation of the optimal control solution we will distinguish explicitly between controllable and uncontrollable exogenous variables. So, let exogenous vector u be divided into instrument variables (again denoted by u) and uncontrollable exogenous variables, denoted by d. It is assumed that d is known for all time t, either by observation or by anticipation. Therefore the lag structure of d will not be shown explicitly in the model equation. The ARX(p,q)-model will be represented as

$$\begin{aligned} y(t) = {} & \tilde{A}_0 y(t) + \tilde{A}_1 y(t-1) + \tilde{A}_2 y(t-2) + \dots + \tilde{A}_p y(t-p) + \\ & \tilde{B}_0 u(t) + \dots + \tilde{B}_q u(t-q) + \tilde{F} d(t) + \tilde{M} v(t) \end{aligned} \tag{5.1}$$

The transformation of (5.1) to state-space form is accomplished via a stacking procedure, by which a vector $x(t)$, $x:\Omega \times T \rightarrow R^n$ will be constructed as a stacked vector of (delayed) endogenous variables and instrument variables or their combinations. Several ways how to organize the vector x will be discussed.

One preliminary step is required. Assume that $(I-\tilde{A}_0)$ in (5.1) is nonsingular; then (5.1) can be transformed to the reduced form

$$\begin{aligned} y(t) = {} & A_1 y(t-1) + \dots + A_p y(t-p) + \\ & B_0 u(t) + \dots + B_q u(t-q) + Fd(t) + Mv(t) \end{aligned} \tag{5.2}$$

$$\begin{aligned} \text{where } A_i &= (I-\tilde{A}_0)^{-1}\tilde{A}_i, \quad i = 1,2,\dots,p, \\ B_j &= (I-\tilde{A}_0)^{-1}\tilde{B}_j, \quad j = 0,1,\dots,q, \\ F &= (I-\tilde{A}_0)^{-1}\tilde{F}, \\ M &= (I-\tilde{A}_0)^{-1}\tilde{M}. \end{aligned}$$

We will provide three propositions how to transform (5.2) into state-space form. Two of them are well-known in the econometric literature.

<u>Proposition 5.1 (Chow, 1975, p. 153)</u>

The reduced form (5.2) and the first-order reduced form

$$x(t+1) = Ax(t) + Bu(t+1) + \bar{F}d(t+1) + \bar{M}v(t+1) \tag{5.3a}$$

$$y(t) = Cx(t) \tag{5.3b}$$

are equivalent in the sense of representing identical input-output descriptions, for appropriately related initial conditions, where

$$x(t) := [y(t);\ y(t-1);\dots;y(t-p+1);\ u(t);\dots;u(t-q+1)] \in R^{pk+qm}$$

$$A := \left[\begin{array}{ccccc|ccccc} A_1 & \cdot & \cdot & \cdot & A_p & B_1 & \cdot & \cdot & \cdot & B_q \\ & & & & 0 & & & & & \\ & & & & \cdot & & & & & \\ & I & & & \cdot & & & \emptyset & & \\ & & & & \cdot & & & & & \\ & & & & 0 & & & & & \\ \hline & & & & & 0 & \cdot & \cdot & \cdot & 0 \\ & & & & & & & & & \cdot \\ & & \emptyset & & & & & I & & \cdot \\ & & & & & & & & & \cdot \\ & & & & & & & & & 0 \end{array}\right] \qquad B := \left[\begin{array}{c} B_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}\right]$$

$$\bar{M} := [M;0;\ldots;0],\ \bar{F} := [F;0;\ldots;0],\ C := [I,0,\ldots,0].$$

Proof By definition of $x(t)$, the proof is immediate.

□

Remarks

1. Since in (5.3a) the control vector is represented by $Bu(t+1)$ instead of $Bu(t)$, (5.3) is not a Gaussian system representation as in (2.3); therefore it is named a first-order reduced form.
   However, $x(t)$ can be recognized as the state, and the representation (5.3) is suitable for control applications as shown in Chow (1975).
2. The output equation (5.3b) does not depend on $u(t)$; such models are called proper (vs. improper).

The second transformation that we will propose requires a preliminary step and a definition. Let $s := \max(p,q)$ and rewrite (5.2) as

$$\begin{aligned} y(t) = {} & A_1 y(t-1) + \ldots + A_s y(t-s) + \\ & B_0 u(t) + \ldots + B_s u(t-s) + Fd(t) + Mv(t) \end{aligned} \tag{5.4}$$

where

$$A_{p+1} = \ldots = A_s = 0 \quad \text{if } s = q \text{ and}$$
$$B_{q+1} = \ldots = B_s = 0 \quad \text{if } s = p.$$

Define the backward shift operator L by

$$Ly(t+1) = y(t).$$

The forward shift operator is then L^{-1} : $L^{-1}y(t) = y(t+1)$.

Proposition 5.2 (Aoki, 1976, p. 26)

The reduced form (5.2) and the state-space representation

$$x(t+1) = Ax(t) + Bu(t) + \bar{F}d(t+1) + \bar{M}v(t+1) \tag{5.5a}$$
$$y(t) = Cx(t) + Du(t) \tag{5.5b}$$

are equivalent in the sense of representing identical input-output descriptions, for appropriately related initial conditions, where

$$x(t) := [z_1(t);\dots;\ z_s(t)] \in R^{k*\max(p,q)}$$

with $z_i(t)$, $i = 1,2,\dots,s$ defined in the proof,

$$A := \begin{bmatrix} A_1 & & \\ A_2 & & \\ \cdot & & \\ \cdot & I & \\ \cdot & & \\ A_s & 0 \dots & 0 \end{bmatrix} \quad B := \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{B}_s \end{bmatrix} \quad \bar{M} := \begin{bmatrix} M \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \bar{F} := \begin{bmatrix} F \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$C := [I,\ 0,\dots,0],\ D := B_0,\ \hat{B}_i := A_i B_0 + B_i,\ i = 1,2,\dots,s.$$

Proof. To obtain (5.5) from (5.2), define $z_1(t) := y(t) - B_0 u(t)$ and substitute this into (5.4):

$$\begin{aligned} z_1(t) &= A_1 y(t-1) + \dots + A_s y(t-s) + \\ &\qquad B_1 u(t-1) + \dots + B_s u(t-s) + Fd(t) + Mv(t) = \\ &L[A_1 y(t) + B_1 u(t) + \\ &\quad L[A_2 y(t) + B_2 u(t) + \dots + \\ &\qquad\qquad L[A_s y(t) + B_s u(t)]\dots]] + Fd(t) + Mv(t) \end{aligned}$$

or

$$z_1(t+1) = A_1 y(t) + B_1 u(t) + z_2(t) + Fd(t+1) + Mv(t+1)$$

by multiplication with L^{-1} on both sides and by appropriate definition of $z_2(t)$. Elimination of $y(t)$ yields:

$$z_1(t+1) = A_1 z_1(t) + z_2(t) + \hat{B}_1 u(t) + Fd(t+1) + Mv(t+1).$$

Proceeding in this way recursions for $z_2(t),\dots,z_s(t)$ can be derived, and the stacking of the $z_i(t)$ into $x(t)$ completes the construction of (5.5).

Conversely, to obtain (5.2) from (5.5), use (5.5b):

$$y(t) = Cx(t) + Du(t) \text{ or}$$
$$z_1(t) = y(t) - B_0u(t)$$

By substitution into (5.5a) one obtains expressions for $z_2(t),\ldots,z_s(t)$ and eventually (5.4) or equivalently (5.2). □

Remark. (5.4) can be rewritten as

$$\begin{aligned} y(t) - B_0u(t) = {} & A_1[y(t-1) - B_0u(t-1)] + \ldots + \\ & A_s[y(t-s) - B_0u(t-s)] + \\ & (B_1 + A_1B_0)u(t-1) + \ldots + (B_s + A_sB_0)u(t-s) + \\ & Fd(t) + Mv(t) \end{aligned}$$

As in the proof of Proposition 5.2, define $z_1(t) = y(t) - B_0u(t)$ and construct a state-space form with the state

$$x(t) := [z_1(t);z_1(t-1);\ldots;z_1(t-s+1);u(t-1);\ldots;u(t-s+1)].$$

A state-space form arises of similar structure as (5.3), but with a control term $Bu(t)$ instead of $Bu(t+1)$. Using the procedure of Proposition 5.1 it is possible to obtain (improper) state-space forms, and not only first-order reduced forms. □

Proposition 5.2 has been attributed here to Aoki (1976), but is also known from many other and earlier references. In fact it is the multivariable generalization of Kelvin's method which dates back to 1876; see Kailath (1980) for the case of continuous-time, proper systems.

In both methods (Propositions 5.1 and 5.2) we did not take the structure of $A_0,\ldots,A_p,B_0,\ldots,B_q$ into account. We will now show how we can modify the first-order reduced form (5.3) by omitting the zero columns in the matrices $A_0,\ldots,A_p$, $B_0,\ldots,B_q$.

The main result is Proposition 5.5. Some definitions are given first.

Definition 5.3

$\tilde{A}_i$, the $k \times k_i$-matrix which arises by omitting zero columns of A_i, $i = 2,3,\ldots,p$

$\tilde{B}_j$, the $k \times k_{j+p}$-matrix which arises by omitting zero columns of B_j, $j = 1,2,\ldots,q$

$S_i \in R^{k\times k_i}$, selector matrices, defined implicitly by $\tilde{A}_i = A_i S_i$, $i = 1,\ldots,p$

$S_{p+j} \in R^{k\times k_{p+j}}$, selector matrices, defined implicitly by $\tilde{B}_j = B_j S_{p+j}$, $j = 1,\ldots,q$

$I_i \subset \{1,2,\ldots,k\}$ and $I_{p+j} \subset \{1,2,\ldots,m\}$ are index sets, associated with S_i and S_{p+j} respectively, and consisting of the row numbers of their unit vectors.

Example 5.4. For some fixed i, let $A_i = \begin{bmatrix} * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{bmatrix}$,

where * denotes an arbitrary nonzero entry of A_i.
Clearly $k = 3$, $k_i = 2$ and $S_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Now $I_i = \{1,3\}$ which corresponds with the "active" elements of y. Note that $S_i^T S_i = I_{k_i}$ □

In order to modify the Chow realization (5.3), another condition must be fulfilled which will now be stated.

Given the matrices $A_1,\ldots,A_p$, $B_1,\ldots,B_q$, construct the reduced matrices $\tilde{A}_p$ and $\tilde{B}_q$, as in definition 5.3 with corresponding index sets I_p and I_{p+q}. Then proceed by constructing $\tilde{A}_{p-1}$ and $\tilde{B}_{q-1}$, but retain the zero columns that are needed to fulfil the condition

$$I_{p-1} \supseteq I_p \text{ and } I_{p+q-1} \supseteq I_{p+q}$$

If we continue in this way for $\tilde{A}_{p-i}$, $\tilde{B}_{p-j}$, $i = 2,\ldots,p-2$, $j = 2,\ldots,q-1$, we say that the nesting property for an ARX(p,q)-model holds if

$$\begin{aligned}&\text{a) } I_2 \supseteq I_3 \supseteq \cdots \supseteq I_p \text{ and}\\ &\text{b) } I_{p+1} \supseteq I_{p+2} \cdots \supseteq I_{p+q}\end{aligned} \tag{5.6}$$

Let us apply the above construction to the matrices A_i, $i = 1,\ldots,p$, B_j, $j = 0,\ldots,q$ of an arbitrary ARX(p,q)-model, while retaining the right zero columns. Under this proviso, we may conclude that the nesting property for the index sets holds.

This fact will be used to modify the stacking procedure of Proposition 5.1.

<u>Proposition 5.6</u>. Assume that the nesting property (5.6) holds. The reduced form (5.2) and the modified first-order reduced form, given by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t+1) + \bar{F}d(t+1) + \bar{M}v(t+1)\\ y(t) &= Cx(t) \end{aligned} \tag{5.7}$$

are equivalent in the sense of representing identical input-output descriptions, where

$$x(t) := [y(t);y^{(2)}(t-1);\ldots;y^{(p)}(t-p+1);u^{(1)}(t);\ldots;u^{(q)}(t-q+1)]$$

$$y^{i}(t) := S_i^T y(t),\ u^{(j)}(t) := S_{p+j}^T u(t)$$

$$A := \left[\begin{array}{cccc:ccc} A_1 & \tilde{A}_2 & \cdots & \tilde{A}_p & \tilde{B}_1 & \cdots & \tilde{B}_q \\ S_2^T & & & 0 & & & \\ & P_3 & & \vdots & & \emptyset & \\ & & & \vdots & & & \\ & & P_p & 0 & & & \\ \hline & & & & 0 & \cdots & 0 \\ & & & & P_{p+2} & & \vdots \\ & \emptyset & & & & & \vdots \\ & & & & & P_{p+q} & 0 \end{array}\right] \quad B := \left[\begin{array}{c} B_0 \\ 0 \\ \vdots \\ 0 \\ \hline S_{p+1}^T \\ 0 \\ \vdots \\ 0 \end{array}\right]$$

$$C := [I,0,\ldots0],\ \bar{M} := [M;0;\ldots;0],\ \ \bar{F} := [F;0;\ldots;0]$$

P_i is defined by $y^{(i)} = P_i y^{(i-1)}$ and $y^{(i)}(t) = S_i^T y(t)$, $i = 3,\dots,p$. P_{p+j} is defined by $u^{(j)} = P_{p+j} u^{(j-1)}$ and $u^{(j)}(t) = S_{p+j}^T u(t)$, $j = 2,\dots,q$.

Proof. By omitting the zero columns in A_i and B_j (5.2) can be rewritten as

$$\begin{aligned} y(t) = {} & A_1 y(t-1) + \tilde{A}_2 y^{(2)}(t-2) + \dots + \tilde{A}_p y^{(p)}(t-p) + \\ & B_0 u(t) + \tilde{B}_1 u^{(1)}(t-1) + \dots + \tilde{B}_q u(q)(t-q) + \\ & Fd(t) + Mv(t). \end{aligned}$$

Since the nesting property is assumed to hold, there exist matrices P_i such that

$$y^{(2)} = S_2^T y, \quad y^{(i)} = P_i y^{(i-1)}, \quad i = 3,\dots,p$$

$$u^{(1)} = S_{p+1}^T u, \quad u^{(j)} = P_{p+j} u^{(j-1)}, \quad j = 2,\dots,q.$$

By definition of x(t) and the stacking procedure, the proof follows.

$\square$

In the next example we will illustrate how the application of Proposition 5.5 will reduce the dimension of the state vector (in comparison with Proposition 5.1), and what is the interpretation of the nesting property.

Example 5.6. Consider an ARX(3,0)-model in reduced form with k = 3 and m = 1 of the following form (set F = 0, M = 0)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{13} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{t-1} + \begin{bmatrix} a_{21} & 0 & 0 \\ a_{22} & 0 & 0 \\ a_{23} & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{t-2} + \begin{bmatrix} 0 & a_{32} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{t-3}$$

$$+ \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} u(t)$$

The index sets are $I_1 = \{1,2,3\}$, $I_2 = \{1\}$, $I_3 = \{2\}$ and the nesting property will hold if $y_2(t-2)$ is 'activated', i.e. included in the state vector:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_1(t-1) \\ y_2(t-1) \\ y_2(t-1) \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{21} & 0 & a_{32} \\ 0 & a_{12} & 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{13} & a_{23} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-1) \\ y_2(t-1) \\ y_3(t-1) \\ y_1(t-2) \\ y_2(t-2) \\ y_2(t-3) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

The nesting property can be interpreted in terms of the coefficients of the ARX-model. If some endogenous variable or instrument has a nonzero coefficient for delay d, then the zero columns for delay d-1, d-2,...,1 corresponding with that variable, cannot be omitted.

Note that the procedure of Proposition 5.1 would result in a 3*3 + 0*1 = 9-dimensional state vector; Proposition 5.2 implies a 3*max(3,0) = 9-dimensional state vector too.

□

<u>Remark</u>

The transformation from ARX(p,q)-model to state-space form is closely related to the following theory. Let $y:T \to R^k$ and $u:T \to R^m$ and define the autoregressive, moving average or ARMA(p,q)-model

$$\begin{aligned} &A_0y(t) + A_1y(t-1) + \ldots + A_py(t-p) = \\ &B_0u(t) + B_1u(t-1) + \ldots + B_qu(t-q) \end{aligned}$$

In <u>deterministic realization theory</u> one analyses the problem how to find an equivalent (in input-output sense) deterministic, linear system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

such that the dimension of the state x(t) is minimal. The corresponding system is called a <u>minimal realization</u>. This problem has been reviewed by Kailath (1980), ch. 6. The connection with Proposition 5.5 will not be explored any further.

Conclusion

In this section we have presented three ways to transform an ARX-model into state-space form. These forms are suitable for control applications, although not all of them can be recognized as Gaussian system representations as given in (2.3).

The resulting state vector consists of (combinations of) endogenous and exogenous variables. By suitable manipulation the dimension of the state vector can be reduced, which is attractive for computational purposes. The theoretical implications for the dimension of x will be explored in section 5.3.4.

5.2.2. The optimal control problem for ARX-models and Gaussian system representations

In this section we will confront two ways to formulate the optimal control problem. First, we consider the ARX-model, the transformation to state-space form and the resulting control problem. Secondly, we formulate the control problem directly for a Gaussian system representation. It is our aim to recognize the differences.

A. Optimal control for the ARX-model

From the shared on-line model data $\{y(s), u(s), d(s), s \leq t\}$ an ARX(p,q)-model can be estimated. In reduced form we have

$$y(t) = A_1 y(t-1) + A_2 y(t-2) + \dots + A_p y(t-p) + B_0 u(t) + \dots + B_q u(t-q) + Fd(t) + Mv(t) \tag{5.8}$$

The decision makers are supposed to formulate target variables as

$$z(t) = Hy(t) + Ju(t) \tag{5.9}$$

Using the results of the previous section we know that a state x(t) can be defined, such that

$$x(t) = f[y(t),\ y(t-1),\ldots,\ u(t),\ u(t-1),\ldots]. \qquad (5.10)$$

where the function f depends on the particular stacking procedure that has been chosen. Then (5.8) can be transformed into an improper state-space representation of the following form:

$$x(t+1) = Ax(t) + Bu(t) + Fd(t+1) + Mv(t+1) \qquad (5.11a)$$

$$y(t) = Cx(t) + Du(t) \qquad (5.11b)$$

with $z(t) = Hy(t) + Ju(t)$

Due to the construction of x(t) in (5.10), and the fact that the on-line model data comprise $\{y(s),\ u(s),\ s \leq t\}$, we notice that x(t) will be observed by the decision makers. In this complete-state observation case (section 2.5) we choose as class of admissible strategies the class of feedback control laws $\underline{U}$.

Now substitute (5.9) into (5.11b) and recall the cost function (2.8), then the resulting optimal control problem is formulated as

$$\underset{u \in \underline{U}}{\text{minimize}}\ E[J(u)]$$

$$\text{subject to} \quad x(t+1) = Ax(t) + Bu(t) + Fd(t+1) + Mv(t+1) \qquad (5.12a)$$

$$z(t) = HCx(t) + (J+HD)u(t) \qquad (5.12b)$$

This is a linear-quadratic Gaussian optimal control problem for controlled variables z(t) in the complete-state observation case.

B. Optimal control for the Gaussian system representation

For convenience we repeat the Gaussian system representation (section 2.5) and include the uncontrollable exogenous variables d.

$$x(t+1) = Ax(t) + Bu(t) + Fd(t) + Mv(t) \qquad (5.13a)$$

$$y(t) = Cx(t) + Du(t) + Nv(t) \qquad (5.13b)$$

$$z(t) = Hx(t) + Ju(t) \qquad (5.13c)$$

Note that there is a slight inconsistency in the notation of $Fd(t) + Mv(t)$, compared to (5.11a). This does not affect the generality of the result, since $d(t)$ is assumed to be known for all $t \in T$ and $(v(t), t \in T)$ is assumed to be white noise.

From section 2.5 we repeat that the interpretation of $y(t)$ is different from the one given for the ARX-model. The decision makers observe the system's state through $(y(t), t \in T)$, afflicted by measurement noise $Nv(t)$.

In this partial-state observation case we define as class of admissible strategies the class of closed-loop strategies $\underline{U}$ (see section 2.5).

The optimal control problem can be stated as

$$\underset{u \in \underline{U}}{\text{minimize}} \quad E[J(u)] \text{ subject to (5.13)}$$

This is a LQG-problem for controlled variables $z(t)$ in the partial-state observation case. It is different from the problem formulated for the ARX-case, since the stochastic assumptions underlying (5.13) are different from the ones underlying (5.8).

Let us attempt to unify the two problems in one formal representation. To that aim we require the solution of the LQG-problem in the partial-state observation case (see Appendices 5B, 5C and 5D). The solution to the LQG-problem can be split in two parts. First, $x(t)$ is estimated recursively based on the measurements $\{y(0), y(1), \ldots, y(t-1)\}$. The optimal estimate is denoted by $\hat{x}(t)$, the so-called Kalman filter. Secondly, the optimal control problem is restated in terms of the filtered state $\hat{x}(t)$. The main result is now that the control problem can be solved independently from the filter problem. Hence we can conceive the control problem as one of complete-state observation, i.e. in the state $\hat{x}(t)$.

Define the class of feedback control laws $\underline{U}$ for the filtered system with state $\hat{x}(t)$. A recursion for $\hat{x}(t)$ is provided in Appendix 5B. Then the resulting control problem can be stated as

$$\underset{u \in \underline{U}}{\text{minimize}} \quad E[J(u)]$$

$$\text{subject to} \quad \hat{x}(t+1) = A\hat{x}(t) + Bu(t) + Fd(t) + K\varepsilon(t) \tag{5.14a}$$

$$z(t) = H\hat{x}(t) + Ju(t) \tag{5.14b}$$

Also from Appendix 5B it is known that $(\varepsilon(t), t \in T)$ is a Gaussian, white noise process. Hence (5.12) and (5.14) are formally equivalent.

Conclusion

We have formulated optimal control problems for ARX-models and for Gaussian system representations. In case of the ARX-model we are led to a complete-state observation control problem, whereas in the case of Gaussian system representation a partial-state observation control problem arises. However, if the state estimator $\hat{x}(t)$ is interpreted as the new, observable state, the two optimal control problems have the same format, viz.

$$\underset{u \in \underline{U}}{\text{minimize}} \quad E[J(u)]$$

$$\text{subject to} \quad x(t+1) = Ax(t) + Bu(t) + Fd(t) + Mv(t)$$

$$z(t) = Hx(t) + Ju(t) \tag{5.15}$$

This problem formulation is particularly convenient for application, as we will show later on. A number of theoretical results have been invoked in order to achieve this result. It must be shown that the solutions to the LQG-problem can be split into two stages; the recursion for $\hat{x}(t)$, i.e. (5.15a) must be derived, and the control problem in terms of $\hat{x}(t)$ must be solved. These topics will be pursued in the Appendices 5B, 5C and 5D.

5.3. Properties of Gaussian system representations

In this section we will explore the qualitative and quantitative behaviour of a Gaussian system representation (5.13). Since the Gaussian system representation is a very useful tool in various fields of applications its properties will be analysed. Attention is restricted to econometric applications. In particular we will concentrate on the relation between the instrument variables and the state and the relation between the state and the target variables. These two relationships are governed by the concepts of controllability and observability respectively. These concepts determine whether a realization is minimal or not. This fact has practical consequences, since the state dimension governs the computational performance of the optimal control algorithm. This fact has conceptual consequences as well, since a nonminimal realization has undesirable properties as will be pointed out.

The quantitative relation between the instrument variables and the target variables provides insight into the effectiveness of the instrument variables; moreover, it sets the stage for application of model reduction techniques. We will mention a possible approach to model reduction, which seems fruitful for the type of econometric models considered here.

A broad spectrum of topics is treated in this section. The properties of qualitative nature of the Gaussian system representation are dealt with in the first four subsections (stability, controllability, observability and minimality); the quantitative relation between instrument variables and target variables is treated in the section on multipliers.

5.3.1. Stability

Consider the evolution of the state x(t) in (5.13) and assume that no exogenous input and noise affect the state. We call this the unforced system

$$x(t+1) = Ax(t) \tag{5.16}$$

Suppose (5.16) is a model of an economic system, e.g. a national economy. If the economy is unaffected by exogenous variables and noise, how will it behave in the long run? This question will only be answered for the <u>model</u> (5.16) which is at best an abstraction of the real economic system.

Let us first introduce some notation and definitions. The set of eigenvalues of A, sp(A), is called the spectrum of A. An eigenvalue of A is called stable if $|\lambda| < 1$, marginally stable if $|\lambda| = 1$ and unstable in other cases. The matrix A is called stable if all eigenvalues of A are stable. The following result is well-known:

> The unforced system (5.16) is called asymptotically stable, defined as $\lim_{t\to\infty} x(t) = 0$ for any initial condition, iff A is stable.

It is hard to assess whether in reality an economic system is stable or not. Perhaps it is unstable, but stabilized by the decision maker. Anyhow, the location of the spectrum of A in the complex plane, will reveal the qualitative, unforced behaviour of (5.16) which may be stable or unstable.

5.3.2. Controllability

In this section we will investigate the relation between the instrument variables u and the state x. For reasons of convenience in exposition we omit the uncontrollable exogenous variables and the noise. Hence, we set F = 0, M = 0, N = 0 in (5.13), and consider the linear, deterministic, time-invariant, discrete-time system representation, given by

$$x(t+1) = Ax(t) + Bu(t) \tag{5.17a}$$

$$y(t) = Cx(t) + Du(t) \tag{5.17b}$$

where $x:T \to R^n$ is the state, $u:T \to R^m$ the input, $y:T \to R^k$ the output. Let $T = \{0,1,\dots\}$, $X = R^n$ the state space, $\underline{U}$ the class of admissible strategies. For all $u \in \underline{U}$, $x \in X$, the solution of (5.17) is denoted as

$$x(t,t_0,\ x_0;\ u),$$
$$y(t,t_0,\ x_0;\ u)$$

when the system starts at $x_0 = x(t_0)$.

Definition 5.7

A state $x(t_1) = x_1$ of (5.17) is said to be controllable from a state x_0, if there exists an input $u \in \underline{U}$ such that x_0 can be transferred to x_1 within finite time, or

$$\exists u \in \underline{U},\ \exists t_0 < t_1 \text{ such that } x_1 = x(t_1, t_0, x_0; u)$$

The system (5.17) is said to be controllable if every state $x \in X$ is controllable from every state $x_0 \in X$.

□

A useful characterisation is provided by the following proposition.

Proposition 5.8

The system representation (5.17) is controllable iff

$$\text{rank}[B, AB, \ldots, A^{n-1}B] = n$$

Proof. Padulo and Arbib, 1974, p. 223; Hautus, 1970.

□

Remark. Definition 5.7 is directly specified for a linear, time-invariant system representation. Our definition of controllability is often referred to as reachability. Of course, it does not depend on the output equation (5.17b). The interested reader may consult Kalman, Falb and Arbib, 1969, ch. 2, Hautus (1970), Chen, 1984, ch. 5, Desoer, 1970, ch. 7, Kwakernaak and Sivan, 1972, ch. 6 for a variety of various related concepts.

5.3.3. Observability

As in section 5.3.2, we restrict attention to (5.17) in order to investigate the relation between the state x and the output y. The key question is: suppose the output y is available, the input u is available, can the initial state $x(0)$ be discovered? Note that knowledge of $x(0)$ implies knowledge of all $x(t)$, since the input is known. This question motivates the following definition.

Definition 5.9. Consider (5.17). Let the state $x_0 \neq 0$. The state $x(t_0) = x_0$ is said to be unobservable, if there exists a $t_1 < \infty$ such that the zero input sequence, $u(t) = 0$ for all $t \in [t_0, t_1]$, renders a zero output sequence, $y(t, t_0, x_0; u = 0) = 0$, for all $t \in [t_0, t_1]$.
The system representation (5.17) is said to be observable if no state $x \neq 0$ $(x \in X)$ is unobservable.

□

A test for the observability of (5.17) is given by the following proposition. Note that the notation $(A_1; A_2) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ is used (see page vi).

Proposition 5.10

The system representation (5.17) is observable iff

$$\text{rank}[C; CA; \dots ; CA^{n-1}] = n$$

Proof. Padulo and Arbib, 1974, p. 263; Hautus, 1970

□

The definition of observability is applicable if $x(t)$ cannot be observed, but $y(t)$ does. From section 5.2.2, however, we concluded that we should consider the system representation (5.15), displaying the relation between the state and the target variables. Again for $F = 0$ and $M = 0$, we have

$$x(t+1) = Ax(t) + Bu(t) \tag{5.18a}$$
$$z(t) = Hx(t) + Ju(t) \tag{5.18b}$$

By analogy of definition 5.9 we will call (5.18) observable, if, given z and u, it is possible to discover the initial state x_0. The following example shows that this analogy makes sense.

Example 5.11

Let $x = (x_1; x_2)$ and A, B, H in (5.18) are partitioned conformably, such that (5.15) has the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

$$z(t) = [H_1, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t + Ju(t)$$

Due to the imposed structure on A and H, the target behaviour of z(t) cannot be influenced by $x_2(t)$. For a given control $u \in \underline{U}$, a given target path z is consistent with infinitely many state trajectories x_2. A representation consisting of $(x_1(t), z(t))$ alone, will do equally well.

□

Analogously to Proposition 5.10 one can prove.

Proposition 5.12

The system representation (5.18) is observable iff

$$\text{rank}[H; HA; \ldots ; HA^{n-1}] = n$$

□

5.3.4. Minimal realizations

In order to perceive the potential of controllability and observability we will define the concept of a minimal realization.

Let us consider the state representation (5.17) and assume that $x(0) = 0$. Denote by Y(z) and U(z) the z-transform of the output y(t) and the input u(t), respectively. By taking the z-transform of (5.17) and by elimination of the state x, we obtain the transfer-function representation

$$Y(z) = H(z)U(z)$$

where $H(z) = C(zI-A)^{-1}B + D$ is the transfer function of (5.17). It is said that the internal description of the system has been transformed to an external description. Now consider the converse transformation, where a transfer function $H(z)$ is given and a state-space representation or realization must be found with the same input-output description. Suppose that the resulting state vector has dimension n. Then we say that the realization is minimal if, for any other realization of $H(z)$ with state vector $\tilde{x}$ and $\dim(\tilde{x}) = \tilde{n}$, we have $\tilde{n} \geq n$. (Compare this with the remark at the end of section 5.2.1).

A characterisation of a minimal realization is given in the following theorem.

Theorem 5.13

Given the system representation (5.17). (5.17) is a minimal realization, iff (5.17) is controllable and observable.

Proof. Desoer, 1970, p. 181.

□

By the analogy between (5.17) and (5.18) we observe that Theorem 5.13 also holds for (5.18).

For computational and theoretical reasons, there is a need to work with minimal realizations. The optimal control algorithms, to be presented in section 5.4, will reveal that the effort in computation strongly depends on the dimension of the state. Theoretically we observe that, along the lines of definitions 5.8 and 5.12, a nonminimal realization has two drawbacks: first, a part of the state cannot be controlled by the input and, secondly, different state trajectories may lead to one target path.

This section has a connection with section 5.2.1 (the transformation of ARX-models) as well: The Propositions 5.1, 5.2 and 5.5 can be seen as realization methods, attempting to find low-dimensional realizations.

Major complications, however, arise when we try to operationalize the concepts of observability and controllability in a practical setting. The characterisation given in Propositions 5.10 and 5.12 leads to difficult numerical problems: how to determine the rank of a matrix or how to determine the dimension of the subspace of X that cannot be observed (controlled). Therefore a more quantitative notion, e.g. a quantitative measure for controllability and observability is required in practical applications.

Especially in large macroeconometric models we encounter the following situation. A major part of the model consists of definitional and technical relations, and a minor part consists of behavioural equations. The model size can be reduced by elimination (through substitution) of those endogenous variables that do not contribute to the dynamic behaviour or the formulation of the target variables. The resulting reduced model can be investigated on controllability and observability properties. If it turns out that a relatively small part of the model constitutes its dynamic behaviour, the model can be reduced even further by a suitable approximation (in some sense).

The complicated problem of approximation by a lower-order model may be tackled via the controllability and observability properties of the model. These properties, measured in a quantitative way, show which inputs have a (very) small effect on the state, and which states have a very small effect on the target variables. However, a complication arises here, since an input variable can have a major (minor) impact on a part of the state, whereas this part of the state has a minor (major) impact on the target variables. A model approximation technique should acknowledge this fact.

A rigorous, mathematical framework for model approximation along the lines described above has been initiated by Moore (1981), see also Aoki, 1984, ch. 9. We have mentioned this topic since we believe it is of importance for the control approach to macroeconometric models. The application of control techniques invariably leads to time-consuming computations which depend essentially on the dimension of the state vector.

5.3.5. Multipliers for the relation between instrument variables and target variables

In this section we study the quantitative relation between the instrument variables and the target variables in the system representation (5.18). The so-called multipliers will measure the effect of a unit change in $u(t)$ on the target variables $z(t+k)$, $k = 0,1,\dots$.

We can distinguish between dynamic multipliers which reveal the effect of a sustained unit change of the instrument variables, and interim multipliers which measure the effect of one unit impulse of the instrument variables.

In systems and control literature these objects are called step response and impulse response function respectively. The interim multipliers are also known as Markov parameters.

Now consider (5.18).

Define:

$$M_k := \frac{\partial z(t+k)}{\partial u(k)},\ k = 0,1,\dots$$

Then M_0 is called the impact multiplier of u on z, M_k is called the interim multiplier for lag $k = 1,2,\dots$, $\sum_{s=1}^{k} M_s$ is called the dynamic multiplier for lag $k = 1,2,\dots$ and $\sum_{s=1}^{\infty} M_s$ is called the long-run or equilibrium multiplier.

Proposition 5.14

For the system representation

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t),\ x(0) \\ z(t) &= Hx(t) + Ju(t) \end{aligned}$$

with A a stable matrix, there holds

$$\begin{aligned} M_0 &= J \\ M_k &= HA^{k-1}B,\ k = 1,2,\dots \\ M_\infty &= H(I-A)^{-1}B + J \end{aligned}$$

Proof. M_0 and M_k follow by computing $\frac{\partial z}{\partial u}$ in

$$z(t) = HA^t x(0) + \sum_{s=1}^{t-1} HA^{t-s-1} Bu(s) + Ju(t)$$

When A is stable, the following fact holds: $\sum_{i=1}^{\infty} A^i = (I-A)^{-1}$. This relation immediately yields M_∞.

□

Let $z = (z_1; z_2; \ldots; z_r)$ and $(u = u_1; u_2; \ldots; u_m)$. The multiplier between a particular target variable z_i, $i = 1,2,\ldots,r$ and an instrument variable u_j, $j = 1,2,\ldots,m$, is denoted by

$$(M_k)_{ij} := \frac{\partial z_i(t+k)}{\partial u_j(t)}$$

By discriminating between target variables and instrument variables of several decision makers, it is possible to make the impact of DMi's instrument variables on DMj's $(i \neq j)$ target variables explicit. This analysis is useful for comparison and interpretation of cooperative and noncooperative dynamic game solutions.

In addition, the multipliers provide a useful quantitative measure for the overall relation between instrument variables and target variables. By tabulating all multipliers $(M_k)_{ij}$, $i = 1,2,\ldots,r$, $j = 1,2,\ldots,m$ for a number of lags, $k = 0,1,2,\ldots$, it is possible to check which instrument variables do and which do not significantly affect the proposed target variables. For instance, for a fixed instrument variable u_j, this happens when $(M_k)_{ij} < \varepsilon_{ij}$ for all $k = 0,1,2,\ldots$ and $j = 1,2,\ldots,r$.

The size of the ε_{ij} depends on the variables u_i and z_j, i.e. their dimension, mean, variance, scale. When an econometric model is formulated in growth rates, the multipliers are dimensionless which facilitates the analysis. By a multiplier table for all u_i and z_j the complete quantitative coupling between the instrument variables and the target variables is displayed. An example will be given in chapter 8.

5.4. Optimal control algorithms

The optimal control problem, as formulated in section 3.3, will now be solved for the global dynamics, shared information case. We consider the situation that two decision makers act upon the same model and share their observations. The control problem will be solved for a non-cooperative solution concept (the Nash equilibrium concept, definition 3.7b) and a cooperative solution concept (the Pareto concept, definition 3.7c).

The control problems will be tackled via the technique of stochastic dynamic programming. A review of necessary and sufficient conditions for optimality will be presented in Appendix 5A, formulated in terms of the "abstract control system" (Striebel, 1975).

These optimality conditions can be applied to the standard linear-quadratic Gaussian model. Its solution requires the Kalman filter; the formulation of the Kalman filter will be given in Appendix 5B. The linear-quadratic Gaussian control problem for one decision maker will be stated and proved in full detail in Appendix 5C. Properties of the optimal LQG-solution, i.e. the certainty equivalence result and the separation result, will be stated in Appendix 5D.

The results of these four appendices are used to present the Nash equilibrium solution and the Pareto solution in sections 5.4.1 and 5.4.2 respectively. The proof of the theorem for the Nash solution is relegated to Appendix 5E, and can be skipped without loss of continuity.

5.4.1. The dynamic Nash solution

For convenience we restate the stochastic dynamic game with two decision makers, see (4.1). Uncontrollable exogenous variables are included explicitly.

$$x(t+1) = Ax(t) + B_1u_1(t) + B_2u_2(t) + Fd(t) + Mv(t) \tag{5.19a}$$
$$y(t) = Cx(t) + Du(t) + Nv(t) \tag{5.19b}$$
$$z_1(t) = H_1x(t) + J_1u_1(t) \tag{5.19c}$$
$$z_2(t) = H_2x(t) + J_2u_2(t) \tag{5.19d}$$

where

$v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$,
$x(0) \in G(m_0, \Sigma_0)$.

DMi uses his control u_i based on observations y, shared by DM1 and DM2, in order to manipulate his target variables z_i. All system matrices of (5.19) are known to DM1 and DM2. The cost functions for the decision makers are

$$J_i(u_1, u_2) = \sum_{t=t_0}^{t_f-1} \{\|z_i(t) - \bar{z}_i(t)\|^2_{Q_i} + \|u_i(t) - \bar{u}_i(t)\|^2_{R_i}\} +$$

$$x^T(t_f)Q_{if}x(t_f),\ i = 1,2$$

with $(\bar{z}_i(t),\ \bar{u}_i(t),\ t \in [t_0,\ t_f-1])$ the desired target and instrument paths of DMi, i = 1,2.

The function J_i in (5.20) differs from the cost function (3.2). The reason is a mathematical argument, given in Appendix 5C.

As set of admissible control strategies for decision maker DMi, we take the set of closed-loop control laws $\underline{U}_i$, see section 2.5. Due to the sharing of information, we take $\underline{U}_1 = \underline{U}_2$.

For the control problem described above, we shall derive dynamic Nash Equilibrium strategies. The definition of dynamic optimality is a refinement of the usual definition of optimality, i.e. the minimization of expected costs. (See Appendix 5A).

The following notation will be used. The sigma-algebra's generated by y(t) and $\{y(0),\ y(1),\ldots,y(t)\}$ are denoted by $F^{y(t)}$ and F^y_t respectively. The notion of sigma-algebra is required to define conditional expectations with respect to sigma-algebra's, see Ash, 1972, ch. 6. The control system induces a probability measure P_u for a control strategy u; conditional expectation with respect to this measure is denoted by $E_u[\cdot|\cdot]$. The notation u^t is used for a truncated control strategy up to time t, i.e. $u^t = \{u(0),\ldots,u(t)\}$. $u^t\big|_*$ is then equivalent to $(u^t)^* \equiv \{u^*(0),\ldots,u^*(t)\}$.

Definition 5.15

(u_1^*, u_2^*) is a dynamic Nash Equilibrium pair, if for all $t \in T$ the following two conditions hold

1. $E_{u_1^* u_2^*}[J_1(u_1^*, u_2^*) \mid F_{t-1}^y] \leq E_{u_1 u_2^*}[J_1(u_1, u_2^*) \mid F_{t-1}^y]$
   a.s. $P_{u_1^* u_2^*}$, for all $u_1 \in \underline{U}_1$ such that $u_1^t = u_1^t|_*$
2. $E_{u_1^* u_2^*}[J_2(u_1^*, u_2^*) \mid F_{t-1}^y] \leq E_{u_1^* u_2^*}[J_2(u_1^*, u_2) \mid F_{t-1}^y]$
   a.s. $P_{u_1^* u_2^*}$, for all $u_2 \in \underline{U}_2$ such that $u_2^t = u_2^t|_*$

□

Definition 5.15 is a straightforward combination of the Nash definition 3.7b and definition 5A.1. It concerns the situation that the decision makers have reached time t by playing their optimal strategies. Then they have no alternative but continuing these optimal strategies.

We will compute a dynamic Nash equilibrium pair for the model (5.19) with cost functions (5.20) and strategy sets $\underline{U}_i$. The technique will be stochastic dynamic programming. The optimality conditions are stated in Appendix 5A (Theorem 5A.4); the application to the single-decision-maker LQG-problem is stated in Appendix 5C (Theorem 5C.3). In the case of the Nash solution one can think of two LQG-problems, such that DM1's control problems is parametrized by u_2^* and vice versa, and the solution of a pair of equations in u_1^* and u_2^*. These aspects will be clarified in the proof of the following theorem.

Theorem 5.16

Given the Gaussian system representation

$$\begin{aligned}
x(t+1) &= Ax(t) + B_1u_1(t) + B_2u_2(t) + Fd(t) + Mv(t)\\
y(t) &= Cx(t) + Du(t) + Nv(t)\\
z_1(t) &= H_1x(t) + J_1u_1(t)\\
z_2(t) &= H_2x(t) + J_2u_2(t)
\end{aligned}$$

where

$v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$,
$x(0) \in G(m_0,\ \Sigma_0)$,

and cost functions $J_i(u_1,\ u_2)$, $i = 1,2$, given by (5.20), and strategy sets $\underline{U}_1 = \underline{U}_2$.
Assume that the ε-lattice condition of definition 5A.3 holds and that $NVN^T > 0$, $Q_i \geqslant 0$, $R_i > 0$, $Q_{if} \geqslant 0$, $i = 1,2$.
The dynamic Nash equilibrium pair is given by

$$u_1^*(t) = G_1(t)\hat{x}(t) + g_1(t)$$

$$u_2^*(t) = G_2(t)\hat{x}(t) + g_2(t)$$

where

$$G_i(t) = -[R_i + J_i^T Q_i J_i]^{-1}[B_i^T P_i(t+1)E^{-1}(t+1)\tilde{A} + J_i^T Q_i H_i], \quad i = 1,2$$

$$g_i(t) = -[R_i + J_i^T Q_i J_i]^{-1}[B_i^T P_i(t+1)E^{-1}(t+1)\tilde{\pi}(t+1) + \pi_i(t+1)],$$
$i = 1,2$

$$P_i(t) = (H_i + J_i G_i(t))^T Q_i (H_i + J_i G_i(t)) + G_i^T(t) P_i(t+1) G_i(t) + \tilde{A}^T E^{-T}(t+1) P_i(t+1) E^{-1}(t+1)\tilde{A}, \ i = 1,2$$

$$s_i(t) = (H_i + J_i G_i(t))^T Q_i (J_i g_i(t) - \bar{z}_i(t)) + G_i^T(t) R_i (g_i(t) - \bar{u}_i(t)) + \tilde{A}E^{-T}(t+1)P_i(t+1)E^{-1}(t+1)\tilde{\pi}(t+1) + \tilde{A}^T E^{-T}(t+1)s_i(t+1), \ i=1,2$$

$$r_i(t) = (J_i g_i(t) - \bar{z}_i(t))^T Q_i (J_i g_i(t) - \bar{z}_i(t)) + (g_i(t) - \bar{u}_i(t))^T R_i (g_i(t) - \bar{u}_i(t)) + \tilde{\pi}^T(t+1)E^{-T}(t+1)P_i(t+1)E^{-1}(t+1)\tilde{\pi}(t+1) + 2\tilde{\pi}^T(t+1)E^{-T}(t+1)s_i(t+1) + r_i(t+1) + \operatorname{trace}[H_i^T Q_i H_i \Sigma(t) + K^T(t) P_i(t+1) K(t) V_\varepsilon(t)], \ i = 1,2$$

with initial conditions $P_i(t_f) = Q_{if}$

$$s_i(t_f) = 0$$

$$r_i(t_f) = \text{trace}[Q_{if}\Sigma(t_f)],\ i = 1,2$$

The following auxiliary variables have been used

$$\tilde{A} = A-B_1(R_1+J_1^TQ_1J_1)^{-1}J_1^TQ_1H_1 - B_2(R_2+J_2^TQ_2J_2)^{-1}J_2^TQ_2H_2$$

$$E(t+1) = I+B_1(R_1+J_1^TQ_1J_1)^{-1}B_1^TP_1(t+1) + B_2(R_2+J_2^TQ_2J_2)^{-1}B_2^TP_2(t+1)$$

$$\tilde{\pi}(t+1) = Fd(t)-B_1(R_1+J_1^TQ_1J_1)^{-1}\pi_1(t+1) - B_2(R_2+J_2^TQ_2J_2)^{-1}\pi_2(t+1)$$

$$\pi_i(t+1) = B_i^Ts_i(t+1) - R_i\bar{u}_i(t) - J_i^TQ_iH_i\bar{z}_i(t),\ i = 1,2.$$

In addition, the variables $\hat{x}(t)$, $K(t)$, $\Sigma(t)$ and $V_\varepsilon(t)$ have been defined in Theorem 5B.2, by (5.27) - (5.30) respectively.
The optimal cost J_i^*, $i = 1,2$ is given by

$$J_i^* = m_0^TP_i(0)m_0 + 2m_0^Ts_i(0) + r_i(0),\ i = 1,2$$

Proof. See Appendix 5E.

□

Remark

It must be checked that recursive computation of the main variables $\{G_i(s),\ g_i(s),\ P_i(s),\ s_i(s),\ r_i(s),\ s = t_f,\dots,t_0,\ i = 1,2\}$ is possible. The following implication diagram shows the order in which the computations must be performed. The update from final time t_f to time t_f-1 will be shown which can be repeated for all $s = t_f-1,\dots,t_0$.

Note that $\tilde{A}$ can be computed from known matrices, that $\{\bar{z}_i(s),\ \bar{u}_i(s),\ d(s),\ s = t_0,\dots,t_f,\ i = 1,2\}$ are known vectors, and that $\{\Sigma(t),\ K(t),\ V_\varepsilon(t),\ t = t_0,\dots,t_f\}$ can be computed independently by running the Kalman filter (see Appendix 5B).

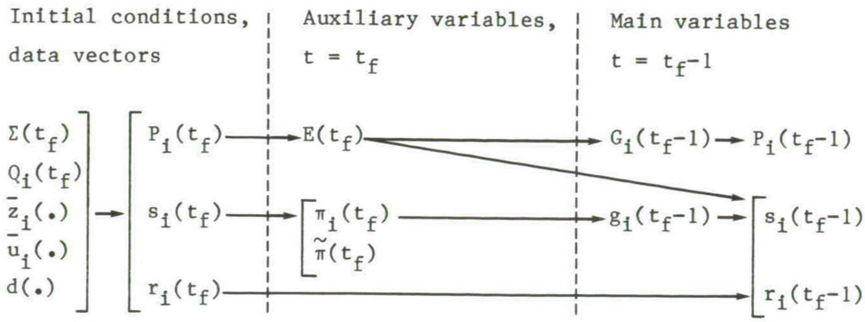


Diagram 5.1. Recursive computation of the Nash solution.

5.4.2. The Pareto solution

Consider again for the global dynamics, shared information case the Gaussian system representation (5.19) with cost functions (5.20) and strategy sets $\underline{U}_1 = \underline{U}_2$.

We will now compute the Pareto solution (definition 3.7c). The Pareto-solution reflects a situation of full cooperation between the decision makers. Therefore it seems, intuitively, possible to stack the controls u_1 and u_2 into one vector u and to solve a combined control problem. This is indeed the case; a rigorous proof for the deterministic case can be found in De Zeeuw, 1984, pp. 74-78. The result which carries over to the stochastic case as presented here is that the set of Pareto solutions can be characterised in the following way.

Let $\alpha \in R$, $0 \leqslant \alpha \leqslant 1$ and define

$$J(u_1,u_2) := \alpha J_1(u_1,u_2) + (1-\alpha)J_2(u_1,u_2).$$

If $J_1(u_1,u_2)$ and $J_2(u_1,u_2)$ are convex functions in their arguments and if $\underline{U}_1$ and $\underline{U}_2$ are convex sets, then the set of Pareto solutions can be uniquely represented as the solution to the minimization of $J(u_1,u_2)$ with respect to $(u_1;\ u_2)$. Note that this solution is parametrized by α.

This result enables us to reformulate the model (5.19) and the costs (5.20) in such a way that the standard LQG-problem for one decision maker arises. Define

$$
\begin{aligned}
u &:= (u_1;\ u_2),\ \bar{u} := (\bar{u}_1;\ \bar{u}_2),\\
z &:= (z_1;\ z_2),\ \bar{z} := (\bar{z}_1;\ \bar{z}_2),\\
Q &:= \mathrm{diag}(\alpha Q_1,\ (1-\alpha)Q_2),\ R := \mathrm{diag}(\alpha R_1,\ (1-\alpha)R_2),\\
Q_f &:= \alpha Q_{1f} + (1-\alpha)Q_{2f}\\
B &:= (B_1,\ B_2),\ J := \mathrm{diag}(J_1,\ J_2)\\
H &:= (H_1;\ H_2)
\end{aligned}
$$

Then (5.19) reduces to

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + Fd(t) + Mv(t)\\
y(t) &= Cx(t) + Du(t) + Nv(t)\\
z(t) &= Hx(t) + Ju(t)
\end{aligned}
\tag{5.21}
$$

The composed cost function, denoted by J(u), becomes

$$
J(u) = \sum_{t=0}^{t_f-1} \{\|z(t)-\bar{z}(t)\|_Q^2 + \|u(t)-\bar{u}(t)\|_R^2\} + x^T(t_f)Q_f x(t_f) \tag{5.22}
$$

Except for the terms $Fd(t)$, $\bar{z}(t)$, $\bar{u}(t)$, this is the standard LQG-problem, stated and solved in Appendix 5C. We can therefore state the Pareto solution immediately.

Theorem 5.17

Given the Gaussian system representation (5.21) with cost function (5.22). Assume that the condition of definition 5A.3 holds, and that $NVN^T > 0$, $Q \geq 0$, $R > 0$, $Q_f \geq 0$.
The Pareto solution is given by

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t^* = -[R+J^TQJ+B^TP(t+1)B]^{-1}[J^TQH+B^TP(t+1)A]\hat{x}(t)
$$

where

$$P(t) = H^TQH + A^TP(t+1)A - [J^TQH+B^TP(t+1)A]^T\cdot$$
$$\cdot[R+J^TQJ+B^TP(t+1)B]^{-1}[J^TQH+B^TP(t+1)A],$$

$$s(t) = A^TP(t+1)Fd(t) + A^Ts(t+1) - H^TQ\bar{z}(t)$$
$$-[J^TQH+B^TP(t+1)A]^T[J^TQJ+R+B^TP(t+1)B]^{-1}\pi(t+1)$$

$$r(t) = r(t+1) + d^T(t)F^TP(t+1)Fd(t) + 2d^T(t)F^Ts(t+1) +$$
$$\bar{u}^T(t)R\bar{u}(t) + \bar{z}^T(t)Q\bar{z}(t)$$
$$-\pi^T(t+1)[J^TQJ+R+B^TP(t+1)B]^{-1}\pi(t+1) +$$
$$\mathrm{trace}[K^T(t)P(t+1)K(t)V_\varepsilon(t) + H^TQH\Sigma(t)]$$

$$\pi(t+1) = B^TP(t+1)Fd(t) - J^TQ\bar{z}(t) - R\bar{u}(t) + B^Ts(t+1)$$

with initial conditions $P(t_f) = Q_f$

$$s(t_f) = 0$$
$$r(t_f) = \mathrm{trace}[Q_f\Sigma(t_f)],$$

and $\hat{x}(t)$, $\Sigma(t)$, $K(t)$, $V_\varepsilon(t)$ are defined in Appendix 5B.
The optimal costs J^* is given by

$$J^* = m_0^TP(0)m_0 + 2m_0^Ts(0) + r(0)$$

Proof. The Pareto solution is obtained by transformation to the LQG-problem. The proof of the LQG-result in Appendix 5C can be traced back word for word, under the proviso that the cost function is of the form (5.22) and that the expected cost-to-go is of the form

$$\hat{x}^T(t)P(t)\hat{x}(t) + 2\hat{x}^T(t)s(t) + r(t)$$

□

Appendix 5A The abstract control system and conditions for optimality

In this appendix we present a mathematical framework for a very general control system; our source is Striebel (1975), where the precise measure-theoretic formulations and the proofs of the theorems can be found. For this type of control system optimality conditions have been stated using martingale theory which seems to be one of the most general (and elegant) approaches to optimal control.

The abstract control system

1. (Ω, F) is a measurable space. $T = \{0, 1, \dots, t_f\}$ the time-index set.
2. $(F_t,\ t \in T)$ and $(G_t,\ t \in T)$ are families of increasing sigma-algebra's, satisfying $F_t \subset F$, $G_0 = \{\emptyset, \Omega\}$, $G_t \subset F_t$ for all $t \in T$.
3. $\underline{U}$ is an abstract index set, and denotes the class of admissible control strategies. A control strategy is denoted by $\{u\} \equiv u = \{u(0), u(1), \dots, u(t_f - 1)\}$, whereas a truncated control strategy (up to time t) is denoted by $\{u^t\} \equiv u^t = \{u(0), \dots, u(t)\}$; $\underline{U}^t$ is the corresponding class of truncated control strategies.
4. A control strategy induces a family of probability measures $P_{\{u^t\}}: F_t \to [0,1]$. Expectation with respect to $P_{\{u\}} = P_u$ will be denoted by E_u. Because the control system is restricted to be causal, the probability measure satisfies the condition: for all $t \in T$, $P_{\{u\}}(A) = P_{\{u^t\}}(A)$ for all events $A \subset F_t$.
5. $J : \Omega \times \underline{U} \to R_+$ denotes the cost function.

The interpretation of the control system is as follows. F_t represents all the available information in the system, G_t is the information of the decisionmaker. The convention $G_0 = \{\emptyset, \Omega\}$ indicates that initially the decision maker knows nothing; since G_t is increasing, he has perfect recall.

For a complete description of the abstract control system, a rigorous definition of all spaces, measurability conditions and consistency requirements, we refer to Striebel, 1975, ch. 1. We will proceed by stating conditions for optimality.

Two definitions of optimality will be made.

Definition 5A.1. A control law $u^* \in \underline{U}$ is said to be dynamic (or conditional) optimal if for each $t \in [0,t_f]$

$$E_{u^*}[J(u)|G_t] \leq E_u[J(u)|G_t] \text{ a.s. } P_{u^*}$$

for all $u \in \underline{U}$ such that $u^t = u^t|_*$.

Definition 5A.2. A control law $u^* \in \underline{U}$ is said to be static optimal if

$$E_{u^*}[J(u^*)] \leq E_u[J(u)] \text{ for all } u \in \underline{U}.$$

Definition 5A.1, due to Striebel (1975), states that the controller, having arrived at $t \in T$, has no alternative but proceeding optimally, since any other control will increase his costs. Definition 5A.2 is a special case of definition 5A.1 (set $t = 0$ and use $G_0 = \{\emptyset, \Omega\}$) and is the familiar and natural notion of minimization of the expected costs.

The relation between the two types of optimality can be explored further by the following technical definition.

Definition 5A.3. The ε-lattice condition for the abstract control system holds, if, whenever there exist $u_i \in \underline{U}$, $i = 1,2$, with $u_1^t = u_2^t$ for some $t \in T$, then for any $\varepsilon > 0$ there exists $u_3 \in \underline{U}$ with $u_i^t = u_3^t$, $i = 1,2$ and $E_{u_3}[J(u_3)|G_t] \leq E_{u_i}[J(u_i)|G_t] + \varepsilon$ a.s. P_{u_i}, $i = 1,2$.

This is a technical condition (see Striebel, 1975, Appendix A), which can be interpreted as the richness of the class of control strategies. It will be used when we state the optimality conditions below. In addition, one can prove that for Gaussian system representations like (5.24), but formulated in continuous-time setting, definition 5A.2 also implies definition 5A.1 (Kumar and Van Schuppen, 1981). Hence the two definitions are equivalent in this case, and it is conjectured that this is also true for the discrete time case. In the sequel we will only consider dynamic optimal control laws.

In deterministic, dynamic programming the optimality of u^* could be rephrased in terms of the value function based upon the concept of the cost-to-go. In the stochastic framework for dynamic optimal control laws the conditional loss function $V_t : \Omega \times \underline{U}^t \to R_+$ plays a similar role. This function, understood to be G_t-measurable, will simply be denoted as $V_t(u)$, and will be used to formulate necessary and sufficient conditions for optimality.

Finally, let the class of admissible controls be such that $J(u) < \infty$ a.s. P_u for all $u \in \underline{U}$. Necessary and sufficient conditions for optimality are summarized below.

Theorem 5A.4. Suppose the condition of definition 5A.3 holds for the abstract control system.
Then u^* is dynamic optimal, iff
there exists a G_t-measurable process $V_t(u)$,
$V_t : \Omega \times \underline{U}^t \to R_+$ and a $u^* \in \underline{U}$ such that

1. $E_u[V_{t+1}(u)|G_t] \geq V_t(u)$ a.s. P_u, for all $u \in \underline{U}$, for all $t \in T$ (5.23)
2. $E_u[J(u)|G_{t_f}] = V_{t_f}(u)$ a.s. P_u, for all $u \in \underline{U}$ (5.24)
3. $E_{u^*}[V_{t+1}(u^*)|G_t] = V_t(u^*)$ a.s. P_{u^*}, for all $t \in T$ (5.25)

Proof. Striebel, 1975, lemma 4.3.3 and theorem 4.3.8.

□

Remarks

1. If the ε-lattice condition in Theorem 5A.4 is dropped, then the statement of the theorem is only sufficient.
2. Condition (1) in Theorem 5A.4 claims that $V_t(u)$ is a submartingale for all $u \in \underline{U}$ and condition (3) that it is a martingale for $u = u^*$.

Appendix 5B The Kalman filter

In this appendix we present for a Gaussian system representation without inputs the Kalman filter.

Definition 5B.1. A Gaussian system representation without inputs can be formulated as

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t) \\ y(t) &= Cx(t) + Nv(t) \end{aligned} \tag{5.26}$$

where

$x : \Omega \times T \to R^n$, $y : \Omega \times T \to R^k$, $v : \Omega \times T \to R^\ell$

$v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$,

$x(0) \in G(m_0, \Sigma_0)$; $T = \{0,1,\ldots\}$ is the time-index set.

$\square$

Note that in (5.26), $(x(t),\ t \in T)$ is a Gauss-Markov process and $(y(t),\ t \in T)$ is a Gaussian process (see e.g. Jazwinski, 1970).

The Kalman filter

Let $\hat{x}(t) := E[x(t) | F^y_{t-1}]$ be the conditional expectation of the state $x(t)$, given the sigma-algebra generated by the observations $\{y(0),\ldots,y(t-1)\}$.

Let $\Sigma(t) := E[(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T | F^y_{t-1}]$ be the conditional error covariance.

Recursive expressions for $\hat{x}(t)$ and $\Sigma(t)$ are provided by the following theorem.

Theorem 5B.2

Given the Gaussian system representation of definition 5B.1.
Assume that $NVN^T > 0$.
$\hat{x}(t)$ is generated by the Kalman filter

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + K(t)[y(t)-C\hat{x}(t)] \\ \hat{x}(0) &= m_0 \end{aligned} \tag{5.27}$$

where

$$K(t) = [A\Sigma(t)C^T + MVM^T][C\Sigma(t)C^T + NVN^T]^{-1}, \tag{5.28}$$

$$\Sigma(t+1) = A\Sigma(t)A^T + MVM^T - [A\Sigma(t)C^T + MVN^T]\cdot$$
$$[C\Sigma(t)C^T + NVN^T]^{-1}[A\Sigma(t)C^T + MVN^T]^T, \tag{5.29}$$

$$\Sigma(0) = \Sigma_0$$

Let $\varepsilon(t) := y(t) - C\hat{x}(t)$, then $\varepsilon(t) \in G(0, V_\varepsilon(t))$, with

$$V_\varepsilon(t) = C\Sigma(t)C^T + NVN^T \tag{5.30}$$

and $(\varepsilon(t), t \in T)$ is white. The process $(\varepsilon(t), t \in T)$ is called the innovations process.

□

From a statistical viewpoint, the Kalman filter can be interpreted as a recursive, least-squares estimator $\hat{x}(t)$ of $x(t)$. It can be proved that the estimator is unbiased and has minimum variance.

The original proof stems from Kalman, 1960. We will not repeat his argument, but only exhibit the essential element of the Kalman filter: the update by the innovation $y(t) - C\hat{x}(t)$, which reflects the information that could not be anticipated from previous measurements.

We will assume that all stochastic variables have finite second moments. The stochastic variables can be embedded in a Hilbert space and the Kalman filter can be derived by repeated application of the orthogonal projection lemma (Luenberger, 1969, p. 51). In the example that follows this lemma is only exploited once.

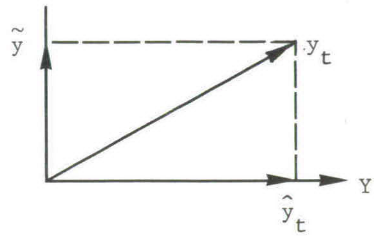
Example 5B.3

Let $\hat{x}$ be the optimal estimate (in least-squares sense) of $x : \Omega \to R^n$ based on a set of measurement Y (e.g., Y is spanned by $\{y(0),\ldots,y(t-1)\}$. Let $\Sigma := E[(x-\hat{x})(x-\hat{x})^T]$ and we want to determine the updated estimate of $\hat{x}$, say $\bar{x}$, if an additional measurement

$$y_t = Cx + w,\ w \in G(0,W)$$

comes available. Let w be independent of x and of Y.

The optimal estimate of y_t based on Y is $\hat{y}_t = C\hat{x}$. Define $\tilde{y} := y_t - C\hat{x}$, the measurement error.



The update of $\hat{x}$ can be seen as a projection on two orthogonal Hilbert spaces, spanned by $\tilde{y}$ and Y. By the orthogonal projection lemma we find for the update $\bar{x}$ of $\hat{x}$:

$$\bar{x} = \hat{x} + E[x\tilde{y}^T]E[\tilde{y}\ \tilde{y}^T]^{-1}\tilde{y}$$

Using the statistics for x and w it is easily shown that

$$\bar{x} = \hat{x} + \Sigma C^T[C\Sigma C^T + W]^{-1}(y - C\hat{x})$$

and that

$$E[(x-\bar{x})(x-\bar{x})^T] = \Sigma - \Sigma C^T[C\Sigma C^T + W]^{-1}C\Sigma$$

In the structure of the last two expressions we recognize (5.27) and (5.29). The argument in this example can be made dynamic, with a little effort, to obtain the result of Theorem 5B.2.

□

Appendix 5C The LQG-problem

In this appendix we will state and prove the linear-quadratic-Gaussian control problem for the partial-state observation case with one decision maker. The conditions for optimality (Theorem 5A.4) and the Kalmen filter (Theorem 5B.2) will be used.

The solution will be presented in a number of steps. First, we will simplify the notation by putting $(\bar{z}(t), \bar{u}(t) = 0, t \in [t_0, t_f])$ in the cost function (2.5). Secondly, we will transform the cost function (2.5) to a form that is suitable for application of the optimality conditions. This form equals the standard format in optimal control (Kwakernaak and Sivan, 1972; Bertsekas, 1976). Thirdly, it must be shown that the Kalman filter equations apply to a linear Gaussian system representation with inputs. This result is based upon a fundamental lemma (Lemma 5C.2). Finally, we are able to solve the LQG-problem (Theorem 5C.3).

For convenience we repeat the LQG-problem. The Gaussian system representation (2.3) together with (2.4) is:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t) \\ y(t) &= Cx(t) + Du(t) + Nv(t) \\ z(t) &= Hx(t) + Ju(t) \end{aligned} \tag{5.31}$$

where

$v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$, $x(0) \in G(m_0, \Sigma_0)$; $T = \{0,1,\ldots,t_f\}$ is the time-index set.

The cost function is

$$J(u) = \sum_{t=t_0}^{t_f} \{z^T(t)Qz(t) + u^T(t)Ru(t)\} \tag{5.32}$$

The class of admissible control strategies is the class of closed-loop control laws $\underline{U}$. The LQG-problem can be formulated as

$$\underset{u \in \underline{U}}{\text{minimize}}\ E[J(u)] \text{ subject to (5.31)}$$

This formulation of the LQG-problem is not the standard formulation as occurs in the literature. Usually in (5.31) the matrix J is set to zero, and the cost function (5.32) has a slightly different form. We will transform the cost function (5.32) to obtain this form. Because the computations are rather tedious we will illustrate the transformation by means of a simple example. The procedure can easily be generalized to the case of (5.31).

Example 5C.1

Let $x : T \to R^n$, $z : T \to R^n$, $u : T \to R^m$ and define the deterministic, linear system

$$x(t+1) = Ax(t) + Bu(t) \tag{5.33a}$$
$$z(t) = Hx(t) + Ju(t) \tag{5.33b}$$

with cost function

$$J(u) = \sum_{t=t_0}^{t_f} \{z^T(t)Qz(t) + u^T(t)Ru(t)\} \tag{5.34}$$

We will show how to minimize J(u) with respect to $u = \{u(0),\ldots,u(t_f)\} \in \underline{U}$. Substitution of (5.33b) into (5.34) yields

$$J(u) = \sum_{t=t_0}^{t_f} \{(Hx+Ju)^TQ(Hx+Ju) + u^TRu\}_t =$$

$$\sum_{t=t_0}^{t_f} (x^TQ_2x + u_2^TR_2u_2)_t \tag{5.35}$$

with $Q_2 := H^TQH - S^T(R+J^TQJ)^{-1}S$

$R_2 := R + J^TQJ$

$S := J^TQH$

$u_2 := u + R_2^{-1}Sx$

(5.33a) can be transformed to

$$x(t+1) = (A-BR_2^{-1}S)x(t) + Bu_2(t) \tag{5.36}$$

The minimization of (5.34) subject to (5.33) has been transformed to the linear-quadratic control problem in standard form: the minimization of (5.35) subject to (5.36).

Now it can be seen that $u_2(t_f)$ does not effect the states $x(t_0),\ldots,x(t_f)$. The minimization of (5.35) implies that

$$u_2^*(t_f) = 0, \text{ or}$$

$$u^*(t_f) = -R_2^{-1}J^TQHx(t_f) \tag{5.37}$$

Hence, it remains to determine $u(t_0),\ldots,u(t_f-1)$. This can be done by dynamic programming (see Kwakernaak and Sivan, 1972).

□

Consider again (5.32) and (5.31). By an identical argument as in Example 5C.1 we can determine the optimal control $u^*(t_f)$. When we omit $u(t_f)$ from (5.32), the standard quadratic cost function with a terminal cost term arises (see Kwakernaak and Sivan, 1972; Bertsekas, 1976)

$$J(u) = \sum_{t=t_0}^{t_f-1} \{z^T(t)Qz(t)+u^T(t)Ru(t)\} + x^T(t_f)Q_fx(t_f) \tag{5.38}$$

The cost function (5.38) will be used from now on. In this formulation the Nash and Pareto solution have been derived (sections 5.4.1 and 5.4.2 respectively).

For the LQG-problem dynamic optimal control laws will be derived, using the optimality conditions of Theorem 5A.4. Note that it is required to rederive the Kalman filter, since we now consider a system representation with inputs, while (5.26) has been stated without inputs. Therefore filter equations are required for the model (5.31).

The derivation of filter equations for a system with inputs will be presented. The fact that we consider a model with inputs implies that the observation sigma-algebra F_t^y of (5.31) depends on the control process. This dependence will be denoted by $F_t^{y,u}$. In order to characterize $F_t^{y,u}$, some notation will be introduced.

Let

$$x_1(t+1) = Ax_1(t) + Mv(t), \; x_1(0) \in G(0, \Sigma_0) \tag{5.39a}$$

$$y(t) = Cx(t) + Nv(t) \tag{5.39b}$$

and

$$x_2(t+1) = Ax_2(t) + Bu(t), \; x_2(0) = m_0 \tag{5.40a}$$

$$y_2(t) = Cx_2(t) + Du(t) \tag{5.40b}$$

such that

$$x(t) = x_1(t) + x_2(t) \tag{5.41a}$$

$$y(t) = y_1(t) + y_2(t) \tag{5.42a}$$

The essential part of the following lemma is due to Wonham (1968). It has also been proven by Fleming and Rishel (1975, p. 191). Both references deal with the continuous-time case. The discrete-time case will be considered here.

<u>Lemma 5C.2</u>

$$F_t^{y,u} = F_t^{y_1}$$

<u>Proof.</u> The proof will be by induction.

For $t = 0$, $G_0 = \{\emptyset, \Omega\}$ and $F_0^{y,u} = F_0^{y_1}$, since $u(0)$ is a deterministic constant, see Appendix 5A.

Assume that $F_{t-1}^{y,u} = F_{t-1}^{y_1}$

From (5.41b) and (5.40b)

$$y_1(t) = y(t) - C_2x_2(t) - Du(t)$$

Now $x_2(t)$ is $F_{t-2}^{y,u}$-measurable by (5.40), $u(t)$ is $F_{t-1}^{y,u}$-measurable by the definition of $\underline{U}$, hence

$$F^{y_1(t)} \subset F_t^{y,u}$$

Using the induction hypothesis, this implies that

$$F_t^{y_1} = F_{t-1}^{y_1} \vee F^{y_1(t)} \subset F_t^{y,u}$$

Conversely, $y(t) = y_1(t) + y_2(t)$ by (5.41b) and since $x_2(t)$, hence $y_2(t)$, is $F_{t-1}^{y,u}$-measurable and by the induction hypothesis also $F_{t-1}^{y_1}$-measurable, we have

$$F^{y(t),u(t)} \subset F_t^{y_1}$$

This implies

$$F_t^{y,u} = F_{t-1}^{y,u} \vee F^{y(t),u(t)} \subset F_t^{y_1}, \text{ hence the result.}$$

□

Now consider (5.41a) and take conditional expectation with respect to $F_{t-1}^{y,u}$. Lemma 5C.2 yields

$$\hat{x}(t) := E[x(t)|F_{t-1}^{y,u}] = E[x_1(t)|F_{t-1}^{y,u}] + x_2(t) =$$

$$\hat{x}_1(t) + x_2(t), \tag{5.42}$$

since $x_2(t)$ is $F_{t-1}^{y,u}$-measurable and $\hat{x}_1(t)$ can be generated by the Kalman filter (Appendix 5B). The expression (5.27) of Theorem 5B.2, (5.40) and (5.42) yield

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K(t)[y(t)-C\hat{x}(t)-Du(t)] \tag{5.43}$$

Analogously to the Kalman filter without inputs, (5.43) can be rephrased in socalled <u>innovations representation</u>

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K(t)\varepsilon(t) \tag{5.44a}$$
$$y(t) = C\hat{x}(t) + Du(t) + \varepsilon(t) \tag{5.44b}$$

with $\varepsilon(t) \in G(0,V_\varepsilon(t))$, $V_\varepsilon(t) := C\Sigma(t)C^T + MVM^T$, $(\varepsilon(t),\ t \in T)$ is white.

Based on the result of Lemma 5C.2, we are able to restate the Kalman filter for a Gaussian system representation with inputs. We will now solve the LQG-problem. In the sequel we will omit the explicit dependence of u in $F_t^{y,u}$.

Theorem 5C.3. Given the Gaussian system representation (5.31) with cost function (5.38). Assume that the ε-lattice condition of definition 5A.3 holds and that $Q \geq 0$, $R > 0$, $Q_f \geq 0$, $NVN^T > 0$, and let $\underline{U}$ be the class of admissible closed-loop strategies.

The dynamic optimal control law $u = u^*(t)$ satisfies

$$
\begin{aligned}
u^*(t) &= L(t)\hat{x}(t)\\
L(t) &= -[R+J^TQJ+B^TP(t+1)B]^{-1}[B^TP(t+1)A+H^TQJ]\\
P(t) &= A^TP(t+1)A + H^TQH - [B^TP(t+1)A+H^TQJ]^T\cdot\\
&\quad [R+J^TQJ+B^TP(t+1)B]^{-1}[B^TP(t+1)A+H^TQJ]\\
r(t) &= r(t+1) + \mathrm{trace}[K^T(t)P(t+1)K(t)V_\varepsilon(t)+H^TQH\Sigma(t)]
\end{aligned}
$$

with initial conditions

$$P(t_f) = Q_f,\ r(t_f) = \mathrm{trace}[Q_f\Sigma(t_f)].$$

$\hat{x}(t)$ is given by (5.27) in Appendix 5B, $K(t)$, $\Sigma(t)$, $V_\varepsilon(t)$ are given by (5.28), (5.29), (5.30) respectively. The expected optimal cost J^* is given by

$$
\begin{aligned}
J^* = m_0^TP(0)m_0 &+ \sum_{j=0}^{t_f-1} \mathrm{trace}[K^T(j)P(j+1)K(j)V_\varepsilon(j)+H^TQH\Sigma(j)]\\
&+ \mathrm{trace}[Q_f\Sigma(t_f)].
\end{aligned}
$$

Proof. Let $V_t(u)$ be the conditional loss function, given the evolution of the process up to time t. Hence $V_t(u)$ can be split in past costs and expected cost-to-go, denoted as $V_t^e(u)$, viz.

$$V_t(u) = E_u[\sum_{j=0}^{t-1} (z^TQz+u^TRu)_j | F_{t-1}^y] + V_t^e(u) \tag{5.45}$$

We will prove by backward induction in time that $V_t^e(u)$ is quadratic in $\hat{x}(t)$, say of the form

$$V_t^e(u) = \hat{x}^T(t)P(t)\hat{x}(t) + r(t) \tag{5.46}$$

Hence, the induction hypothesis at time t+1 is

$$\begin{aligned} V_{t+1}(u) = {} & E_u[\sum_{j=0}^{t} (z^TQz + u^TRu)_j | F_t^y] \\ & + \hat{x}^T(t+1)P(t+1)\hat{x}(t+1) + r(t+1) \end{aligned} \tag{5.47}$$

We will employ the conditions (1), (2) and (3) of Theorem 5A.4, i.e. (5.23), (5.24) and (5.25) to make the following steps.
First, we start the induction at $t = t_f$ by defining $V_{t_f}(u)$. (5.24) will yield the initial conditions for the P(t)- and r(t)-recursions.
Secondly, we substitute (5.47) into (5.23) and perform the minimization with respect to u. The optimal control $u = u^*(t)$ will prove to be linear in $\hat{x}(t)$.
Thirdly, we show that under $u = u^*(t)$, $V_t^e(u)$ is quadratic in $\hat{x}(t)$; from (5.25) the recursions in P(t) and r(t) follow.
Fourthly, the optimal expected costs follow from (5.46) for t = 0.

The procedure set out above, will now be performed.
Define

$$\begin{aligned} V_{t_f}(u) = {} & E_u[\sum_{j=0}^{t_f-1} (z^TQz+u^TRu)_j | F_{t_f-1}^y] + \\ & \hat{x}^T(t_f)P(t_f)\hat{x}(t_f) + r(t_f) \end{aligned} \tag{5.48}$$

Apply (5.24) to (5.38) and (5.48) and use the fact

$$E[x^T(t)Qx(t)|F_{t-1}^y] = \hat{x}^T(t)Q\hat{x}(t) + \mathrm{trace}[Q\Sigma(t)] \tag{5.49}$$

to obtain the initial conditions $P(t_f) = Q_f$
$r(t_f) = \mathrm{trace}[Q_f\Sigma(t_f)]$

Now perform the induction step.
Apply (5.23) to (5.47) and (5.45), simplify the inequality by elimination of the past costs up to time t-1, and use the smoothing property of conditional expectation to yield

$$E_u[(z^TQz+u^TRu)_t + \hat{x}^T(t+1)P(t+1)\hat{x}(t+1) + r(t+1)|F^y_{t-1}] \geq V^e_t(u) \tag{5.50}$$

Substitute $\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K\varepsilon(t)$ and $z(t) = Hx(t) + Ju(t)$ and use (5.49) again, such that (5.50) becomes

$$\hat{x}^T(t)[H^TQH+A^TP(t+1)A]\hat{x}(t) + 2u^T(t)[J^TQH-B^TP(t+1)A]\hat{x}(t)$$

$$+ u^T(t)[R+J^TQJ+B^TP(t+1)B]u(t) + \mathrm{trace}[K^T(t)P(t+1)K(t)V_\varepsilon(t) +$$

$$H^TQH\Sigma(t)] + r(t+1) \geq V^e_t(u)$$

Define

$$u^*(t) = -[R+J^TQJ+B^TP(t+1)B]^{-1}[B^TP(t+1)A+J^TQH]\hat{x}(t)$$

and use the completion-of-the-square formula in u(t), to obtain

$$[u-u^*(t)]^T[R+J^TQJ+B^TP(t+1)B][u-u^*(t)] +$$

$$\hat{x}^T(t)[H^TQH+A^TP(t+1)A - [B^TP(t+1)A+J^TQH]^T.$$

$$.[R+J^TQJ+B^TP(t+1)B]^{-1}[B^TP(t+1)A+J^TQHJ]\hat{x}(t)$$

$$+ \mathrm{trace}[K^T(t)P(t+1)K(t)V_\varepsilon(t) + H^TQH\Sigma(t)] + r(t+1) \geq V^e_f(u)$$

The left-hand-side of this inequality is strict convex in $(u-u^*(t))$; hence the minimum is attained for $u = u^*(t)$.
Note that $u^*(t)$ is linear in $\hat{x}(t)$, and by (5.25) the expected cost-to-go is quadratic in $\hat{x}(t)$, say of the form (5.46).
By comparing the coefficients of the quadratic terms in $\hat{x}(t)$ and the constants the recursions for P(t) and r(t) follow, and are given in the statement of the proof.

The optimal expected cost is $J^* =$
$\hat{x}^T(0)P(0)\hat{x}(0) + r(0) = m_0^T P(0) m_0 + r(0)$.
Computation of r(0) yields the expression in the statement of the proof.

□

Appendix 5D Properties of the solution of the LQG-problem

In this appendix we will explore the behaviour of the Gaussian system representation (5.31) under the optimal solution ($u^*(t)$, $t \in [t_0,t_f]$) given in Theorem 5C.3, and its use in practical application.

Two topics will be discussed. First, we discuss the certainty equivalence result and the separation result (cf. section 4.3.2.2). Secondly, we deal with the statistical properties of the target variables, when the optimal instrument path ($u^*(t)$, $t \in [t_0,t_f]$) is applied to solve the economic planning problem.

The separation result and the certainty equivalence result

We follow the presentation as in Bar-Shalom and Tse, 1974. The certainty equivalent control law $u_{CE}(t)$ for the LQG-problem is obtained as follows. First, compute the optimal deterministic feedback control law for the problem under consideration (i.e. without process noise and in the complete-state observation case).

Denote the result by

$$u(t) = \phi_t(x(t))$$

Then, replace $x(t)$ by its estimate

$$\hat{x}(t) = E[x(t)|F^y_{t-1}]$$

to obtain

$$u_{CE}(t) = \phi_t(\hat{x}(t)).$$

The certainty equivalence result holds, when the optimal control $u^*(t)$ for the stochastic control problem has the same format as the deterministic optimal control with $x(t)$ replaced by $\hat{x}(t)$, i.e. when

$$u^*(t) = \phi_t(\hat{x}(t)).$$

For the LQG-problem, it can be shown that the certainty equivalence result holds. The deterministic optimal control law is given by

$$u(t) = L(t)x(t),$$

where $L(t)$ is given in the statement of Theorem 5C.3.
[this can be proven by retracing the steps of the proof of Theorem 5C.3 for the deterministic case or by consulting Bertsekas, 1976].

The stochastic optimal control law is given by the result of Theorem 5C.3 and satisfies

$$u(t) = L(t)\hat{x}(t).$$

Hence, the certainty equivalence result holds and the original LQG-problem may be interpreted to be equivalent with a "complete-state observation problem" with state $\hat{x}(t)$ (see section 5.2.2).

The separation result is in fact a weaker result than the certainty equivalence result. The separation result holds when the stochastic optimal control depends on the data only via $\hat{x}(t)$, i.e.

$$u(t) = \Psi_t(\hat{x}(t))$$

for some function Ψ_t (possibly different from ϕ_t). Obviously the certainty equivalence result implies the separation result. The reverse implication is false; one may consult (Bar-Shalom and Tse, 1974) for a counter-example.

The separation result for the LQG-problem implies a two-stage solution procedure.

First, by means of the Kalman filter the observations are processed to construct an optimal least-squares estimate of the state. Secondly, this optimal estimate is used as input to a linear, memoryless law which produces the desired optimal control. These two steps can be performed separately; in particular the error covariance $\Sigma(t)$, $t \in T$ can be preprocessed, since it does not depend on the control parameters Q, R, $P(t)$ and is nonrandom.

The use of the LQG-solution for planning purposes

Under the optimal feedback $u^*(t) = L(t)\hat{x}(t)$ the estimated state trajectory $(\hat{x}(t),\ t \in [t_0,t_f])$ provides an expression for the evolution of the system's state. The actual realization of this path can only be given when the observations $(y(t),\ t \in [t_0,t_f])$ have been occurred. The statistical properties of this path, i.e. the first and second moment (of a Gaussian process) can be computed in advance. This will be done below.

Consider the Kalman filter expression

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K(t)\varepsilon(t)$$

under the feedback $u(t) = L(t)\hat{x}(t)$

Then

$$\hat{x}(t+1) = (A+BL(t))\hat{x}(t) + K(t)\varepsilon(t) \tag{5.51}$$

The expectation $(m(t),\ t \in T)$ and the covariance $(\Lambda(t),\ t \in T)$ of $(\hat{x}(t),\ t \in T)$ in (5.51) satisfy

$$m(t+1) = (A+BL(t))m(t),\ m(0) \tag{5.52}$$

$$\begin{aligned}\Lambda(t+1) = {} & (A+BL(t))\Lambda(t)(A+BL(t))^T + \\ & K(t)V_\varepsilon(t)K(t)^T\end{aligned} \tag{5.53}$$

$(m(t),\ t \in T)$ is the expected optimal state path and equals the optimal state path generated by the corresponding deterministic optimal control system. Hence, as a consequence of the certainty equivalence result, the planned optimal state trajectory in the deterministic and in the stochastic case do not differ.

The covariance $(\Lambda(t),\ t \in T)$ provides the decision maker a measure for the uncertainty by which the expected optimal state path will be reached. They allow the policy maker to derive confidence intervals for the optimal paths of the target variables. Using the relation $z = Hx + Ju$ and (5.51) - (5.53), the first and second moment of the target variables z under the optimal feedback law are easily obtained.

Appendix 5E Proof of Theorem 5.16

We will consider the stochastic control problem for DM1 parametrized by u_2^*, see definition 5.15.
Let $V_{it}(u_1, u_2)$ be the conditional loss function of DMi, i = 1,2. The conditions (1) - (3) of theorem 5A.4 read for the dynamic Nash solution, specialized for DM1

$$E_{u_1 u_2^*}[V_{1,t+1}(u_1, u_2^*) | F_{t-1}^y] \geq V_{1t}(u_1, u_2^*) \tag{5.54}$$

and equality for $u_1 = u_1^*$,

$$E_{u_1 u_2^*}[J_1(u_1, u_2^*) | F_{t_f-1}^y] = V_{1t_f}(u_1, u_2^*) \tag{5.55}$$

Similar conditions hold for DM2.

Analogously to the proof of Theorem 5C.3, we will prove by backward induction in t that the expected cost-to-go is quadratic in $\hat{x}(t)$.
At first, however, it is required to obtain an expression for $\hat{x}(t)$.
The state estimator $\hat{x}(t) = E[x(t) | F_{t-1}^y]$ can be derived using results from Appendix 5B.
Restate (5.19a) and (5.19b) as

$$x(t+1) = Ax(t) + (B_1, B_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + Fd(t) + Mv(t)$$

$$y(t) = Cx(t) + Du(t) + Nv(t)$$

By Lemma 5B.1, the sigma-algebra F_{t-1}^y does not depend on the input process $u = (u_1; u_2)$.
The Kalman filter for $\hat{x}(t)$ satisfies, cf. (5.44a)

$$\hat{x}(t+1) = A\hat{x}(t) + (B_1, B_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + Fd(t) + K(t)\varepsilon(t) \tag{5.56}$$

$\varepsilon(t) \in G(0, V_\varepsilon(t))$, $(\varepsilon(t), t \in T)$ is white.
$K(t)$ and $V_\varepsilon(t)$ are given by (5.28) and (5.30).

The induction hypothesis for DM1 at time t+1 states that $V_{1,t+1}$ is of the following form.

$$V_{1,t+1}(u_1, u_2^*) = E_{u_1 u_2^*}[\sum_{j=0}^{t} \{\|z_1-\bar{z}_1\|^2_{Q_1} + \|u_1-\bar{u}_1\|^2_{R_1}\}_j | F_t^y]$$

$$+ \hat{x}^T(t+1)P_1(t+1)\hat{x}(t+1) + 2\hat{x}^T(t+1)s_1(t+1) + r_1(t+1) \qquad (5.57)$$

In the sequel of this proof we closely parallel the proof of Theorem 5C.3.
From (5.55) and the expression for $V_{1t_f}(u_1, u_2^*)$, see (5.57), we obtain the initial conditions for $P_1(t)$, $s_1(t)$ and $r_1(t)$.

$P_1(t_f) = Q_{1f}$,
$s_1(t_f) = 0$,
$r_1(t_f) = \text{trace}[Q_{1f}\Sigma(t_f)]$

Substitution of (5.57) into (5.54), simplification by elimination of the past costs, and the smoothing property of conditional expectation, yield for the left-hand-side of (5.54)

$$E_{u_1 u_2^*}[(z_1-\bar{z}_1)^T Q_1 (z_1-\bar{z}_1)_t + (u_1-\bar{u}_1)^T R_1 (u_1-\bar{u}_1)_t +$$

$$\hat{x}^T(t+1)P_1(t+1)\hat{x}(t+1) + 2\hat{x}^T(t+1)s_1(t+1) + r_1(t+1) | F_{t-1}^y]$$

Substitute $z_1(t) = H_1x(t) + J_1u_1(t)$ and (5.56) for $u_2 = u_2^*$, use the property (5.49), then this expression equals

$$\hat{x}^T(t)[H_1^TQ_1H_1+A^TP_1(t+1)A]\hat{x}(t)+u_1^T(t)[R_1+J_1^TQ_1J_1+B_1^TP_1(t+1)B_1]u(t)$$

$$+ 2u_1^T(t)[J_1^TQ_1(H_1\hat{x}(t)-\bar{z}_1)+B_1^TP_1(t+1)[A\hat{x}(t)+B_2u_2^*(t)+Fd(t)] +$$

$$B_1^Ts_1(t+1) - R_1\bar{u}_1] +$$

$$2\hat{x}^T(t)[A^TP_1(t+1)(B_2u_2^*(t)+Fd(t))+A^Ts_1(t+1)-H^TQ_1\bar{z}_1] +$$

$$(B_2u_2^*(t)+Fd(t))^TP_1(t+1)(B_2u_2^*(t)+Fd(t))+2(u_2^*(t)+Fd(t))^Ts_1(t+1)+$$

$$r_1(t+1)+\bar{z}_1^TQ_1\bar{z}_1+\bar{u}_1^TR_1\bar{u}_1+ \text{trace}[K^T(t)P_1(t+1)K(t)V_\varepsilon(t)+H_1^TQ_1H_1\Sigma(t)] \tag{5.58}$$

Note that (5.58) is strict convex in $(u_1-u_1^*(t))$ and by a completion-of-the-square argument or by differentiation with respect to u_1, we obtain (omit the time-index)

$$u_1^*(t) = -[R_1+J_1^TQ_1J_1+B_1^TP_1B_1]^{-1}[B_1^TP_1(A\hat{x}+B_2u_2^*+Fd)$$

$$+ J_1^TQ_1H_1\hat{x}+B_1^Ts_1-R_1\bar{u}_1-J_1^TQ_1H_1\bar{z}_1] \tag{5.59a}$$

Similarly we obtain for the control problem of DM2:

$$u_2^*(t) = -[R_2+J_2^TQ_2J_2+B_2^TP_2B_2]^{-1}[B_2^TP_2(A\hat{x}+B_1u_1^*+Fd)$$

$$+ J_2^TQ_2H_2\hat{x}+B_2^Ts_2-R_2\bar{u}_2-J_2^TQ_2H_2\bar{z}_2] \tag{5.59b}$$

Explicit expressions for $u_i^*(t)$, $i = 1,2$ can be derived as follows. Define:

$$\pi_i(t+1) := B_i^Ts_i(t+1) - R_i\bar{u}_i(t) - J_i^TQ_iH_i\bar{z}_i(t), \; i = 1,2$$

$$\tilde{\pi}(t+1) := Fd(t) - B_1(R_1+J_1^TQ_1J_1)^{-1}\pi_1(t+1)$$

$$- B_2(R_2+J_2^TQ_2J_2)^{-1}\pi_2(t+1),$$

then a computation with (5.56) yields the expression

$$A\hat{x}(t)+B_1u_1^*(t)+B_2u_2^*(t)+Fd(t) = E^{-1}(t+1)[\tilde{A}\hat{x}(t)+\tilde{\pi}(t+1)] \tag{5.60}$$

where

$$\tilde{A} := A - B_1(R_1 J_1^T Q_1 J_1)^{-1} J_1^T Q_1 H_1 - B_2(R_2 + J_2^T Q_2 J_2)^{-1} J_2^T Q_2 H_2$$

$$E(t+1) := I + B_1(R_1 + J_1^T Q_1 J_1)^{-1} B_1^T P_1(t+1)$$

$$+ B_2(R_2 + J_2^T Q_1 J_2)^{-1} B_2^T P_2(t+1).$$

It will be assumed that $E(t+1)$ is nonsingular for all $t \in T$.
From (5.59) and (5.60) explicit expressions for $u_1^*(t)$ and $u_2^*(t)$ follow

$$u_i^*(t) = -[R_i + J_i^T Q_i J_i]^{-1} [B_i^T P_i(t+1) E^{-1}(t+1) [\tilde{A}\hat{x}(t) + \tilde{\pi}(t+1)] +$$

$$J_i^T Q_i H_i \hat{x}(t) + \pi_i(t+1)], \; i = 1,2 \qquad (5.61)$$

Hence we conclude that $u_i^*(t)$ is affine in $\hat{x}(t)$, $i = 1,2$. If we substitute (5.61) into (5.54), it follows that the expected cost-to-go is quadratic in $\hat{x}(t)$ [this ends the induction step].
Recursions for $P_1(t)$, $s_1(t)$, $r_1(t)$ can be found by comparing quadratic and linear terms in $\hat{x}(t)$ and constants. (Only the result is given, in the statement of Theorem 5.16). Optimal expected costs follow from (5.57) for $t = 0$.

□

CHAPTER SIX

THE GLOBAL DYNAMICS, NON-SHARED INFORMATION CASE

6.1. Introduction

In this chapter we will analyse the control problem stated in section 4.2.2, ad 2. The decision makers are supposed to receive non-shared on-line model data, while they are both supposed to know the values of the parameters in the Gaussian system representation

$$x(t+1) = Ax(t) + B_1u_1(t) + B_2u_2(t) + Mv(t) \tag{6.1}$$

$$y_1(t) = C_1x(t) + D_1u_1(t) + N_1v(t) \tag{6.2a}$$

$$y_2(t) = C_2x(t) + D_2u_2(t) + N_2v(t) \tag{6.2b}$$

Examples of information patterns which display the non-sharing of information have been given in definition 3.6 (3.6e and 3.6f). Attention will be restricted to the case that the on-line model data of the decision makers are completely non-shared. DMi is supposed to base his control upon $\eta_t^{(i)}$, with

$$\eta_t^{(i)} = \{y_i(s),\ u_i(s),\ s < t\}\ , \tag{6.3}$$

the on-line model data of DMi. In this form we have obtained the sharpest confrontation to the shared information case.

Because the optimal control problem will turn out to be rather complicated, the presentation of the results is easily obscured by inessential details. Therefore we will omit the variables that are required for application purposes, but do not alter the essence of the optimal control problem. Hence, we set $(\bar{z}_i(t),\ \bar{u}_i(t)) = 0$, $t \in [t_0,t_f]$, $i = 1,2$ in cost functions (5.20), and we set $z_i(t) = x(t)$ for all $t \in T$, $i = 1,2$ in (5.19c,d).

For the optimal control problem to be studied we specify the classes of admissible control laws that are generated by

$u_{it} = g_{it}(\eta_{t-1}^{(i)})$, $i = 1,2$. For ease of reference we denote these classes of closed-loop control laws by $\underline{U}_{ic}$, $i = 1,2$. The solution concepts to be studied are the Nash Equilibrium concept and the Team concept. Because information is now non-shared, the team problem does not reduce to a control problem with one decision maker, and is in fact similar to a Nash problem (in a simpler notation).

For the specification above we will analyse the optimal control problem. Due to the non-sharing of information we face a more difficult task than in chapter 5. This task will be performed as follows.

First, the fundamental reason for the complexity of the control problem will be investigated. We will propose an approach to circumvent this difficulty. It consists of making approximations in some sense (section 6.2).

Secondly, we will consider the two cases in which an approximation will be made.

1. A finite-dimensional linear system will be constructed, that generates a control law u. This system serves as an approximation to the original system and will be called a <u>compensator</u> (section 6.3).
2. A fixed structure on the control strategy will be imposed, viz. the control strategy is a linear function of a part of the state vector. This is called a <u>decentralized state-feedback control strategy</u>, and can be regarded as a static version of the compensator (section 6.4).

6.2. <u>A control problem with non-shared information</u>

Consider the global dynamics, non-shared information case, exemplified by (6.1), (6.2) and (6.3). Let the solution concept be arbitrary, the cost function are $J_i(u_1,u_2)$, $i = 1,2$, and the classes of admissible control laws are $\underline{U}_{ic}$, $i = 1,2$.

We will make a general remark about the structure of the optimal control problem for this set-up. In finding the optimal control solution the decision maker requires an estimate of the state x(t), upon which his control will be based. We run into the difficulty that a common state estimator $\hat{x}(t) = E[x(t)|F_{t-1}^{y}]$ with $y = (y_1;y_2)$ is unavailable (cf. section 5.4.1). Instead, DMi will estimate the state x(t) by

$$\hat{x}_i(t) := E[x(t)|F_{t-1}^{y_i}] \ , \ i = 1,2 \ .$$

The $\hat{x}_i(t)$ will be different and DMi has no access to $\hat{x}_j(t)$, $i \neq j$, $i,j = 1,2$. Hence DMi must estimate $\hat{x}_j(t)$, $i \neq j$, $i,j = 1,2$. These estimates are

$$E[\hat{x}_1(t)|F_{t-1}^{y_2}] \text{ and } E[\hat{x}_2(t)|F_{t-1}^{y_1}] \ .$$

But the argument repeats and DM1 needs to determine $E[E[\hat{x}_1(t)|F_{t-1}^{y_2}]|F_{t-1}^{y_1}]$, with a similar problem for DM2. Finally, estimates of the form

$$E[E[\ldots[x(t)|F_{t-1}^{y_i}]|F_{t-1}^{y_j}]|F_{t-1}^{y_i}]\ldots] \ , \ i \neq j$$

arise, and since the sigma-algebra's involved are supposed to be disjoint for $i \neq j$ the conditional expectations cannot be simplified.

This phenomenon is called 'the second-guessing' or the closure problem (Ho, 1970). No way out of this dilemma is known, unless additional restrictions on the $\underline{U}_{ic}$ are brought in. We will discuss two possibilities.

First, let $u_i(t)$ be a linear function of all past observations $\{y_i(0),\ldots,y_i(t-1)\}$, $i = 1,2$. Under this assumption some results have been obtained (Willman, 1969; Bagchi and Olsder, 1981), but an essential drawback remains. At every time t the weighting coefficients of all past observations must be recalculated. This can only be done via the iterative solution of a set of equations, to be repeated at each $t \in T$.

Secondly, let $u_i(t)$ be generated by a finite-dimensional, linear system of a fixed dimension: the compensator. Its structure is fixed beforehand and its variables are all known to the decision maker. The parameters, however, are free and must be determined by the decision maker. The state vector of the compensator serves as an estimator for the state of the original system, and its properties depend on the choice of its parameters. Compatible with the non-sharing of on-line model data in (6.3), a compensator for DMi may be defined as follows

$$c_i(t+1) = F_i(t)c_i(t) + G_i(t)u_i(t) + K_i(t)y_i(t),\ c_i(0)$$
$$u_i(t) = L_i(t)c_i(t) \tag{6.4}$$

with T the time-index set, $c_i: \Omega \times T \to R^{n_i}$, $i = 1,2$ the state of the compensator for DMi, $i = 1,2$, and u_i and y_i have the same meaning as in (6.1) and (6.2). The unknowns to be determined by DMi are: $c_i(0)$, $\dim(c_i)$ and $F_i(t)$, $G_i(t)$, $K_i(t)$, $L_i(t)$ for all $t \in T$.

The mechanism of (6.4) is as follows. At time t the decision maker knows $u_i(t)$, $y_i(t)$, $c_i(t)$; he computes by (6.4) $c_i(t+1)$, $u_i(t+1)$, receives an observation $y_i(t+1)$ and another cycle can start. Note that the controls $u_i(t)$ are generated by (6.4), and the remaining problem is to determine all the unknowns. This latter problem will be tackled in the next section. Note that the unknowns are matrix-valued, time-varying parameters.

6.3. Compensator problems

6.3.1. The LQG-compensator

The compensator problem has been stated for the multi-agent case. This problem cannot be understood in a sensible way without a fairly exhaustive treatment of the single-decision-maker, LQG-case. This will be done in this section: the LQG-problem will be solved using the compensator approach.

The LQG-problem has been formulated and solved in Appendix 5C. From (5.31) we have the standard Gaussian system representation

$$x(t+1) = Ax(t) + Bu(t) + Mv(t) \tag{6.5a}$$
$$y(t) = Cx(t) + Du(t) + Nv(t) \tag{6.5b}$$

The cost function will be adapted from (5.38) by setting $z(t) = x(t)$ and becomes (take $t_0 = 0$ for simplicity)

$$J(u) = \sum_{t=0}^{t_f-1} (x^TQx+u^TRu)_t + x^T(t_f)Q_fx(t_f) \tag{6.6}$$

$\underline{U}$ denotes the class of closed-loop control laws.

An alternative way to solve the LQG-problem goes as follows. In accordance with (6.4) a possible choice for the compensator of (6.5) is

$$\begin{aligned} c(t+1) &= F(t)c(t) + G(t)u(t) + K(t)y(t),\ c(0) \\ u(t) &= L(t)c(t) \end{aligned} \tag{6.7}$$

The following objects are unknown: dim(c), c(0), (F(t), G(t), K(t), L(t), $t \in T$). The unknowns will be determined by imposing certain desirable properties on (c(t), $t \in T$). In particular, we require c(t) to be an unbiased, minimum-variance estimator of x(t). These two requirements will be elaborated on in the sequel.

Unbiasedness can be accomplished by assuming that dim(c) = dim(x) = n and that c(0) = E[x(0)] = m_0. Now define e(t) := x(t)-c(t), and compute the recursion for the error e (omit the time-dependence of matrices)

$$\begin{aligned} e(t+1) = {} & (A-KC)e(t) + (A-KC-F)c(t) + (B-KD-G)u(t) \\ & + (M-KN)v(t) \end{aligned} \tag{6.8}$$

The unbiasedness requirement E[e(t)] = 0 is fulfilled, if

$$\begin{aligned} F(t) &= A - K(t)C \\ G(t) &= B - K(t)D \end{aligned}$$

With this choice for F(t) and G(t), (6.7) yields

$$\begin{aligned} c(t+1) &= Ac(t) + Bu(t) + K(t)[y(t)-Du(t)-Cc(t)],\ m_0 \\ u(t) &= L(t)c(t) \end{aligned} \tag{6.9}$$

In (6.9) the only unknowns are (K(t), L(t), $t \in T$) called the filter gain and the control gain respectively. Note that the structure of the Kalman filter (see Appendix 5B) and the linear optimal control law (see Appendix 5C) can already be observed in (6.9).

The LQG-problem, i.e. the minimization of (6.6) subject to (6.5), will now be translated into a minimization problem with respect to ($K(t)$, $L(t)$, $t \in T$). These gains will then be determined via a minimum-variance argument.

Define:

The LQG-compensator problem

Find ($K(t)$, $L(t)$, $t \in T$) such that (6.6) is minimized, subject to the two constraints (6.5) and (6.9).

Now we will show how this problem can be translated into a deterministic optimization problem.

Lemma 6.1. The LQG-compensator problem can be reformulated as

$$\underset{(K(t),\, L(t))}{\text{minimize}} \quad E[J(K,L)] = \text{trace}[\tilde{Q}_f \Sigma(t_f)] + \sum_{t=0}^{t_f-1} \text{trace}[\tilde{Q}(t)\Sigma(t)] \tag{6.10}$$

subject to

$$\Sigma(t+1) = \tilde{A}(t)\Sigma(t)\tilde{A}^T(t) + \tilde{M}(t)V\tilde{M}^T(t)$$

$$\Sigma(0) = \text{diag}(0,\Sigma_0) \tag{6.11}$$

where

$$\tilde{A}(t) := \begin{bmatrix} A+BL(t) & K(t)C \\ 0 & A-K(t)C \end{bmatrix}$$

$$\tilde{M}(t) := [K(t)N;\ M-K(t)N]$$

$$\tilde{Q}(t) := \begin{bmatrix} Q+L^T(t)RL(t) & Q \\ Q & Q \end{bmatrix}$$

$$\tilde{Q}_f := \begin{bmatrix} Q_f & Q_f \\ Q_f & Q_f \end{bmatrix}$$

Proof. Define the augmented system for $(c;\ e)_t$ using (6.8) and (6.9).

$$\begin{bmatrix} c(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} A+BL(t) & K(t)C \\ 0 & A-K(t)C \end{bmatrix} \begin{bmatrix} c(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} K(t)N \\ M-K(t)N \end{bmatrix} v(t)$$

Define $\Sigma(t) := E[\begin{bmatrix} c(t) \\ e(t) \end{bmatrix}(c^T(t)e^T(t))]$, then (6.11) follows.
The expected cost $E[J(u)]$ becomes, in terms of $(c;\, e)_t$:

$$E[J(u)] = \sum_{t=0}^{t_f-1} E[(c^T e^T)\begin{bmatrix} Q+L^T RL & Q \\ Q & Q \end{bmatrix} \begin{bmatrix} c \\ e \end{bmatrix}_t] +$$
$$E[(c^T e^T)\begin{bmatrix} Q_f & Q_f \\ Q_f & Q_f \end{bmatrix}\begin{bmatrix} c \\ e \end{bmatrix}_{t_f}]$$

from which (6.10) follows, using the fact that

$$E[x^T Mx] = \mathrm{trace}[ME[xx^T]]$$

Note that $J(u)$ is replaced by $J(K,L)$, since $(K(t),\, L(t),\, t \in T)$ are now the 'control variables'.
From $x(0) \in G(m_0,\, \Sigma_0)$, we have $\Sigma(0) = \mathrm{diag}(0,\, \Sigma_0)$
$c(0) = m_0$

□

The deterministic control problem stated in Lemma 6.1 will be tackled by the matrix minimum principle (see Appendix 6A).
Define the Hamiltonian

$$H(\Sigma(t),\, P(t),\, K(t),\, L(t)) :=$$

$$\mathrm{trace}[\tilde{A}(t)\Sigma(t)\tilde{A}^T(t)P^T(t+1)+\tilde{M}(t)V\tilde{M}^T(t)P^T(t+1) + \tilde{Q}(t)\Sigma(t)] \quad (6.12)$$

where

$P(t) : T \to R^{2n\times 2n}$ is the costate-function.

$$\text{Let } P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix},\; P_{ij}: T \to R^{n\times n},\; \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix},\; \Sigma_{ij}: T \to R^{n\times n} \quad (6.13)$$

and define the auxiliary matrices $E_i \in R^{n\times 2n}$, $i = 1,2$ by $E_1 := (I_n;0)$, $E_2 := (0;I_n)$ with I_n the n×n-unity matrix.

Proposition 6.2. Consider the deterministic minimization problem stated in Lemma 6.1.
Necessary conditions for the optimality of $(K(t), L(t), t \in T)$ are provided by the minimization of the Hamiltonian H. The first-order conditions are

$$\frac{\partial H}{\partial K(t)} = B^T E_1^T P(t+1)\tilde{A}(t)\Sigma(t)E_1 + RL(t)E_1^T\Sigma(t)E_1 = 0 \tag{6.14a}$$

$$\frac{\partial H}{\partial L(t)} = (E_1-E_2)^T P(t+1)\tilde{A}(t)\Sigma(t)E_2C^T +$$

$$(E_1-E_2)^T P(t+1)(E_1-E_2)K(t)NVN^T = 0 \tag{6.14b}$$

where

$$P(t) = \tilde{A}^T(t)P(t+1)\tilde{A}(t) + \tilde{Q}(t)$$
$$P(t_f) = \tilde{Q}_f \tag{6.15}$$

Proof. Using the definitions of E_1 and E_2, the matrices $\tilde{A}(t)$, $\tilde{M}V\tilde{M}^T$ and $\tilde{Q}$ can be restated as

$$\begin{aligned}
\tilde{A}(t) &= E_1AE_1^T + E_2AE_2^T + E_1BL(t)E_1^T + (E_1-E_2)K(t)CE_2^T,\\
\tilde{M}V\tilde{M}^T &= (E_1-E_2)KNVN^T(E_1-E_2)^T + E_2MVM^TE_2^T +\\
&\quad E_2MVN^TK^T(E_1-E_2)^T + (E_1-E_2)KNVM^TE_2^T,\\
\tilde{Q} &= (E_1+E_2)Q(E_1+E_2)^T + E_1L^TRLE_1.
\end{aligned}$$

Substitute these expressions into (6.12) and compute $\partial H/\partial K = 0$, $\partial H/\partial L = 0$ using the differentiation rules for traces with respect to matrices, see Appendix 6B.
The recursion for P(t) follows from Appendix 6A, conditions (A5) and (A6).

□

We will now derive tractable (and familiar) expressions for $(K(t), L(t), t \in T)$ from (6.14).

Proposition 6.3. Given the first-order conditions (6.14a) and (6.14b) and recursions for the state and costate (6.11) and (6.15) respectively. Let

$$K(t) = (A\Sigma_{22}(t)C^T + MVN^T)(C\Sigma_{22}(t)C^T + NVN^T)^{-1} \tag{6.16a}$$

$$L(t) = -(R+B^TP_{11}(t+1)B)^{-1}B^TP_{11}(t+1)A \tag{6.16b}$$

$$\Sigma_{12}(t) = 0 \tag{6.17a}$$

$$P_{11}(t) = P_{12}^T(t) \tag{6.17b}$$

for all $t \in T$, then this choice for $K(t)$, $L(t)$, $\Sigma_{12}(t)$, $P_{11}(t)$, $P_{12}^T(t)$ satisfies (6.11), (6.15) and (6.14).

Proof. Using the decomposition (6.13) for $P(t)$ and $\Sigma(t)$, the first-order conditions (6.14) can be expanded as follows

$$\begin{aligned}&[(B^TP_{11}(t+1)B+R)L(t) + B^TP_{11}(t+1)A]\Sigma_{11}(t) +\\ &[B^TP_{11}(t+1)K(t)C + B^TP_{12}(t+1)(A-K(t)C]\Sigma_{12}^T(t) = 0\end{aligned}$$

$$\begin{aligned}&(P_{11}-P_{12}^T)_{t+1}[(A+BL(t))\Sigma_{12}(t)C^T + K(t)(C\Sigma_{22}C^T+NVN^T)] +\\ &(P_{12}-P_{22})_{t+1}[A\Sigma_{22}(t)C^T+MVN^T-K(t)(C\Sigma_{22}C^T+NVN^T)] = 0\end{aligned}$$

The choice as made in (6.16) and (6.17) makes the left-hand-sides of the expressions above vanish for each $t \in T$.

□

Discussion of the result

We considered the LQG-compensator problem for which the minimum principle provided the necessary conditions (6.14). With a little effort we derived expressions for the filter gain (6.16a) and the control gain (6.16b), in which we recognize the filter gain (5.34) of Appendix 5B and the optimal, linear control law of Appendix 5C, Theorem 5C.3. In combination with the compensator structure (6.9) we conclude that $c(t)$ can be identified with the Kalman filter $\hat{x}(t)$, cf. (5.27).

This observation enables further interpretation. The result $\Sigma_{12}(t) = 0$ says that the filter error $x-\hat{x}$ is uncorrelated with the filter $\hat{x}$, cf. Example 5B.3.

Moreover, from (6.16) we observe that K(t) only depends on $\Sigma(t)$, write K = $K(\Sigma)$, in particular on the error covariance $\Sigma_{22}(t)$; L(t) only depends on P(t+1), write L = L(P), in particular on the cost $P_{11}(t+1)$ incurred from the state of the compensator.

The result $K = K(\Sigma)$, $L = L(P)$ is not immediately obvious, if one regards the first-order conditions (6.14); in (6.14a) and (6.14b) we observe the product $P(t+1)\tilde{A}(t)\Sigma(t)$, implying a coupling between K(t) and L(t), and between P(t+1) and $\Sigma(t)$. It is interesting to see how this coupling vanishes by inserting the conditions (6.17). Consider the first term in (6.14a) and apply $\Sigma_{12} = 0$, $P_{11} = P_{12}^T$, then

$$B_1^T E_1^T P(t+1)\tilde{A}(t)\Sigma(t)E_1 \overset{(6.17)}{\downarrow =} B_1^T P_{11}(t+1)[A+BL(t)]E_1^T\Sigma(t)E_1$$

Now the factor $E_1^T\Sigma(t)E_1$, if nonsingular, cancels in (6.14a) and (6.16b) arises.

Analogously for K(t) from (6.14b)

$$(E_1-E_2)^T P(t+1)\tilde{A}(t)\Sigma(t)E_2C^T \overset{(6.17)}{\downarrow =} (E_1-E_2)^T P(t+1)(E_1-E_2)[A-K(t)C]\Sigma(t)E_2C^T$$

and now the factor $(E_1-E_2)^T P(t+1)(E_1-E_2)$ cancels, and (6.16a) arises. Note that in (6.14a) by applying (6.17) the matrix $\tilde{A}(t)$ reduces to A+BL(t), and in (6.14b) it reduces to A-K(t)C.

Through this mechanism the separation between control and filtering is achieved. In more general, multi-decision-makers problems it is possible to derive first-order conditions as (6.14). The central question is then whether conditions as in (6.17) can be found, such that the control and filter gains are decoupled. This item will be explored shortly.

6.3.2. The deterministic Nash compensator

The same procedure as for the LQG-compensator will now be followed for a control problem with two decision makers. In this section we present the Nash equilibrium solution in case of a deterministic state representation.

Let the state equation satisfy

$$x(t+1) = Ax(t) + B_1u_1(t) + B_2u_2(t) \tag{6.18}$$

with cost functions

$$J_i(u_1,u_2) = \sum_{t=0}^{t_f-1} \{x^TQ_1x+u_i^TR_{ii}u_i+u_j^TR_{ij}u_j\}_t+x^T(t_f)Q_{if}x(t_f) \tag{6.19}$$

$$i,j = 1,2, \; i \neq j$$

Consider the complete-state observation case such that $u_{it} = g_{it}(x(t))$, $i = 1,2$; the class of feedback control laws is denoted by $\underline{U}_i = \underline{U}_{if}$.

The Nash equilibrium solution follows from

$$\begin{aligned} J_1(u_1^*, u_2^*) &\leq J_1(u_1, u_2^*) \text{ for all } u_1 \in \underline{U}_{1f} \\ J_2(u_1^*, u_2^*) &\leq J_2(u_1^*, u_2) \text{ for all } u_2 \in \underline{U}_{2f} \end{aligned} \tag{6.20}$$

Since feedback strategies are considered, we will assume that u_1 and u_2 are generated as follows

$$\begin{aligned} u_1(t) &= L_1(t)x(t) \\ u_2(t) &= L_2(t)x(t) \end{aligned} \tag{6.21}$$

The gain matrices $(L_1(t), L_2(t), t \in T)$ can be computed via the minimum principle.

<u>Proposition 6.4</u>. Given the optimal control problem with state equation (6.18), cost functions (6.19), classes of feedback control laws $\underline{U}_{1f}$ and $\underline{U}_{2f}$ and solution concept (6.20). Assume that $Q_i \geq 0$, $Q_{if} \geq 0$, $R_{ii} > 0$, $i = 1,2$. Assume that (6.21) holds. The optimal control gains $(L_1(t), L_2(t), t \in T)$ satisfy

$$L_1(t) = -R_{11}^{-1}B_1^TP_1(t+1)E^{-1}(t+1)A$$

$$L_2(t) = -R_{22}^{-1}B_2^TP_2(t+1)E^{-1}(t+1)A$$

with

$$E(t+1) = I + B_1R_{11}^{-1}B_1^TP_1(t+1) + B_2R_{22}^{-1}B_2^TP_2(t+1)$$

$$P_1(t) = Q_1 + A^TE^{-T}(t+1)[P_1(t+1)B_1R_{11}^{-1}B_1^TP_1(t+1) +$$
$$P_2(t+1)B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2^TP_2(t+1) + P_1(t+1)]E^{-1}(t+1)A$$

$$P_1(t_f) = Q_{1f}$$

$$P_2(t) = Q_2 + A^TE^{-T}(t+1)[P_2(t+1)B_2R_{22}^{-1}B_2^TP_2(t+1) +$$
$$P_1(t+1)B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^TP_1(t+1) + P_2(t+1)]E^{-1}(t+1)A$$

$$P_2(t_f) = Q_{2f}$$

Proof. Substitute (6.21) into (6.19), replace $J_i(u_1, u_2)$ by $J_i(L_1, L_2)$ to obtain from (6.20) the pair of inequalities

$$J_1(L_1^*, L_2^*) \leq J_1(L_1, L_2^*) \text{ for all } L_1,$$
$$J_2(L_1^*, L_2^*) \leq J_1(L_1^*, L_2) \text{ for all } L_2.$$

$J_i(L_1, L_2)$, $i = 1,2$ is computed by substitution of (6.21) into (6.19) and the definition $X(t) := x(t)x^T(t)$. The result is

$$J_i(L_1, L_2) = \sum_{t=0}^{t_f-1} \text{trace}[(Q_i + L_i^TR_{ii}L_i + L_j^TR_{ij}L_j)_tX(t) +$$
$$\text{trace}[Q_{if}X(t_f)], \; i,j = 1,2, \; i \neq j$$

(6.18) in terms of $X(t)$, using (6.21) becomes

$$X(t+1) = (A+B_1L_1+B_2L_2)X(t)(A+B_1L_1+B_2L_2)^T, \; X(0)$$

Two (parametrized) optimal control problems follow. Consider DM1's problem; the corresponding Hamiltonian satisfies

$$H_1(X(t), P_1(t+1), L_1(t), L_2^*(t)) =$$
$$\text{trace}[(A+B_1L_1+B_2L_2^*)X(t)(A+B_1L_1+B_2L_2^*)_t^TP^T(t+1)] +$$
$$\text{trace}[(Q_1+L_1^TR_{11}L_1+L_2^{*T}R_{12}L_2^*)X(t)],$$

with $P_1 : T \rightarrow R^{3n\times 3n}$ the costate function of DM1.

From the minimum principle (Appendix 6A), condition (6A.7) implies

$$\frac{\partial H}{\partial L_1(t)} = 2B_1^T P_1(t+1)[A+B_1L_1^*+B_2L_2^*]_t X(t) + 2R_{11}L_1^*(t)X(t) = 0$$

From condition (6A.5) the recursion for $P_1(t)$ follows. A similar argument holds for DM2. Explicit expressions for $L_1^*(t)$ and $L_2^*(t)$ can be derived from the two first-order conditions.

□

Note that the structure of the proof is equivalent to the proof of Theorem 5.16. Essentially, a pair of parametrized optimal control problems and a pair of equations in L_1^* and L_2^* have been solved.

The solution of Theorem 5.16 reduces to the solution as presented here, when we make the following simplifications. Make the control problem deterministic by setting $M = 0$, $y_i = x$, $i = 1,2$, simplify the cost function by setting $\bar{z}_i = 0$, $\bar{u}_i = 0$, $i = 1,2$ and the state equation by setting $d = 0$.

6.3.3. The stochastic Team compensator

In this section we will analyse the compensator approach for a stochastic control problem with two decision makers. We consider the following model representing the global dynamics, non-shared information case

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u_1(t) + B_2u_2(t) + Mv(t) \\ y_1(t) &= C_1x(t) + D_1u_1(t) + N_1v(t) \\ y_2(t) &= C_2x(t) + D_2u_2(t) + N_2v(t) \end{aligned} \tag{6.22}$$

with

$x(0) \in G(m_0,\Sigma_0)$, $v(t) \in G(0,V)$, $(v(t),\ t \in T)$ is white and independent of $x(0)$. In addition, we assume that $MVN_i^T = 0$, $i = 1,2$, $N_1VN_2^T = 0$ and that the decision makers are supposed to minimize the common cost function

$$J(u_1,u_2) = \sum_{t=0}^{t_f-1} (x^TQx+u_1^TR_1u_1+u_2^TR_2u_2)_t + x^T(t_f)Q_fx(t_f) \tag{6.23}$$

The case of closed-loop control laws $u_{it} = g_{it}(y_i(0),\ldots,y_i(t-1))$ will be considered, generating classes $\underline{U}_{ic}$, $i = 1,2$.

We will employ the conditions for Person-by-Person Optimal control strategies, (definition 3.7 c2), viz.

$$\begin{aligned} E[J(u_1^*,u_2^*)] &\leq E[J(u_1,u_2^*)] \text{ for all } u_1 \in \underline{U}_{1c} \\ E[J(u_1^*,u_2^*)] &\leq E[J(u_1^*,u_2)] \text{ for all } u_2 \in \underline{U}_{2c} \end{aligned} \tag{6.24}$$

Both decision makers are supposed to generate the control actions by means of compensators of the following form

$$\begin{aligned} c_1(t+1) &= Ac_1(t) + B_1u_1(t) + K_1(t)[y_1(t)-D_1u_1(t)-C_1c_1(t)], c_1(0) \\ u_1(t) &= L_1(t)c_1(t) \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} c_2(t+1) &= Ac_2(t) + B_2u_2(t) + K_2(t)[y_2(t)-D_2u_2(t)-C_2c_2(t)], c_2(0) \\ u_2(t) &= L_2(t)c_2(t) \end{aligned} \tag{6.26}$$

Similarly as in section 6.3.1, we assume that $\dim(c_i) = \dim(x)$, $i = 1,2$ and that $c_i(0) = m_0$, $i = 1,2$. Then DMi, $i = 1,2$ must still determine $(K_i(t), L_i(t), t \in T)$. As in the previous sections, we will replace $J(u_1,u_2)$ in (6.23) by $J(K_1, L_1, K_2, L_2)$ in order to state the <u>Stochastic Team Compensator Problem</u>: find the gains $(K_i(t), L_i(t), t \in T, i = 1,2)$ such that

$$\begin{aligned} E[J(K_1^*, L_1^*, K_2^*, L_2^*)] &\leq E[J(K_1, L_1, K_2^*, L_2^*)] \text{ for all } (K_1,L_1) \\ E[J(K_1^*, L_1^*, K_2^*, L_2^*)] &\leq E[J(K_1^*, L_1^*, K_2, L_2)] \text{ for all } (K_2,L_2) \end{aligned}$$

subject to (6.22), (6.25) and (6.26).

<u>Lemma 6.5</u>. Consider the Stochastic Team Compensator Problem. The optimal gains $(K_i^*(t), L_i^*(t), i = 1,2, t \in T)$ follow from the minimization of

$$E[J(K_1,K_2,L_1,L_2)] = \sum_{t=0}^{t_f-1} \text{trace}[\tilde{Q}(t)\Sigma(t)] + \text{trace}[\tilde{Q}_f\Sigma(t_f)] \tag{6.27}$$

subject to

$$\Sigma(t+1) = \tilde{A}(t)\Sigma(t)\tilde{A}^T(t) + \tilde{M}(t)V\tilde{M}^T(t), \ \Sigma(0) \tag{6.28}$$

where

$$\tilde{A}(t) := \begin{bmatrix} A & B_1L_1(t) & B_2L_2(t) \\ K_1(t)C_1 & A+B_1L_1(t)-K_1(t)C_1 & 0 \\ K_2(t)C_2 & 0 & A+B_2L_2(t)-K_2(t)C_2 \end{bmatrix}$$

$$\tilde{M}(t) := [M;\ K_1(t)N_1;\ K_2(t)N_2]$$

$$\tilde{Q}(t) := \mathrm{diag}(Q,\ L_1^T(t)R_1L_1(t),\ L_2^T(t)R_2L_2(t))$$

$$\tilde{Q}_f := \mathrm{diag}(Q_f,\ 0,\ 0)$$

<u>Proof</u>. From (6.22), (6.25) and (6.26), define the augmented system

$$\begin{bmatrix} x \\ c_1 \\ c_2 \end{bmatrix}_{t+1} = \begin{bmatrix} A & B_1L_1 & B_2L_2 \\ K_1C_1 & A+B_1L_1-K_1C_1 & 0 \\ K_2C_2 & 0 & A+B_2L_2-K_2C_2 \end{bmatrix}_t \begin{bmatrix} x \\ c_1 \\ c_2 \end{bmatrix}_t + \begin{bmatrix} M \\ K_1N_1 \\ K_2N_2 \end{bmatrix}_t v(t)$$

The cost function then satisfies

$$J(u_1,u_2) = \sum_{t=0}^{t_f-1} E[(x^T,c_1^T,c_2^T)\mathrm{diag}(Q,L_1^TR_1L_1,\ L_2^TR_2L_2)\begin{bmatrix} x \\ c_1 \\ c_2 \end{bmatrix}_t\ ]$$

$$+\ E[(x^T,c_1^T,c_2^T)\mathrm{diag}(Q_f,\ 0,\ 0)\begin{bmatrix} x \\ c_1 \\ c_2 \end{bmatrix}_{t_f}\ ]$$

By definition of $\Sigma(t) := E[(x;\ c_1;\ c_2)(x^T,c_1^T,c_2^T)_t]$ (6.27) and (6.28) follow.

□

From this deterministic optimization problem we can derive necessary conditions by invoking the minimum principle.

Proposition 6.6. Consider the Stochastic Team Compensator Problem. If $(K_i^*(t),\ L_i^*(t),\ i = 1,2,\ t \in T)$ is the optimal solution for this problem, then there exists a costate function $P : T \to R^{3n\times 3n}$ such that

$$P(t) = \tilde{A}(t)P(t+1)\tilde{A}^T(t) + \tilde{Q}(t),\quad P(t_f) = \tilde{Q}_f \tag{6.29}$$

and the first-order conditions are

$$\frac{\partial H}{\partial L_1(t)} = B_1^T(E_1+E_2)^T P(t+1)\tilde{A}(t)\Sigma(t)E_2 + R_1L_1(t)E_2^T\Sigma(t)E_2 = 0 \tag{6.30a}$$

$$\frac{\partial H}{\partial L_2(t)} = B_2^T(E_1+E_3)^T P(t+1)\tilde{A}(t)\Sigma(t)E_3 + R_2L_2(t)E_3^T\Sigma(t)E_3 = 0 \tag{6.30b}$$

$$\frac{\partial H}{\partial K_1(t)} = E_2^T P(t+1)\tilde{A}(t)\Sigma(t)(E_1-E_2)C_1^T + E_2^T P(t+1)E_2K_1(t)N_1VN_1^T = 0 \tag{6.30c}$$

$$\frac{\partial H}{\partial K_2(t)} = E_3^T P(t+1)\tilde{A}(t)\Sigma(t)(E_1-E_3)C_2^T + E_3^T P(t+1)E_3K_2(t)N_2VN_2^T = 0 \tag{6.30d}$$

where $E_1 := [I;0;0]$, $E_2 := [0;I;0]$, $E_3 := [0;0;I]$, with $I \equiv I_n$ the n×n unity matrix.

Proof. From (6.27) and (6.28) define the Hamiltonian

$$H(\Sigma(t),\ P(t+1),\ L_1(t),\ L_2(t),\ K_1(t),\ K_2(t)) =$$
$$\operatorname{trace}[\tilde{A}(t)\Sigma(t)\tilde{A}^T(t)P^T(t+1) + \tilde{M}(t)V\tilde{M}^T(t)P^T(t+1) + \tilde{Q}(t)\Sigma(t)].$$

Then the recursion for the costate $P(t)$ and the first-order conditions for $K_i(t)$, $L_i(t)$, $i = 1,2$ follow from application of the minimum principle. The computations that must be performed, parallel those of the proof of Proposition 6.2.

□

From (6.30) together with (6.29) and (6.28) more explicit expressions for $K_i(t)$ and $L_i(t)$ are desired; in particular, we would like to establish a separation result, i.e. $L_i = L_i(P)$, $K_i = K_i(\Sigma)$, cf. the result for the LQG-compensator.

No such expressions can be found for this problem. A heuristic argument will clarify that no separation can exist. Consider the first-order conditions (6.30), and in particular the products $P(t+1)\tilde{A}(t)\Sigma(t)$; let $P(t)$ and $\Sigma(t)$ satisfy a decomposition as in (6.13). Analogously to the reasoning for the LQG-compensator in section 6.3.1, we would like to establish a decoupling in $L_i = L_i(P)$, $K_i = K_i(\Sigma)$. This must be achieved by suitable choice of $\Sigma_{ij}(t)$ and $P_{ij}(t)$, $i,j = 1,2,3$.
More in particular, from (6.30a) and (6.30b), the dependence on $\Sigma(t)$ can be eliminated, if some of the $P_{ij}(t+1)$, $\Sigma_{ij}(t)$ are chosen, such that in (6.30a) the factor $E_2^T\Sigma(t)E_2$ cancels and in (6.30b) the factor $E_3^T\Sigma(t)E_3$. Similar remarks hold for (6.30c and d). If we elaborate this expression in (6.28) and (6.29), it follows readily that the following restrictions must be fulfilled:

$$
\begin{aligned}
&P_{12}^T + P_{22} = 0, \ \Sigma_{12} = \Sigma_{22} = \Sigma_{23} = \Sigma_{33} = \Sigma_{13} \\
&P_{12}^T + P_{23} = 0 \\
&P_{13}^T + P_{23}^T = 0 \\
&P_{13}^T + P_{33} = 0
\end{aligned}
\tag{6.31}
$$

Let us see what (6.31) means for the P(t)-recursion. From (6.29) we can compute

$$
\begin{aligned}
(P_{12}^T+P_{22})_t = {} & L_1^TB_1^T(P_{11}+P_{12})_{t+1}(A+B_1L_1) + \\
& (A+B_1L_1-K_1C_1)^T(P_{12}^T+P_{22})_{t+1}(A+B_1L_1) + \\
& L_1^TB_1^T(P_{13}+P_{23})_{t+1}K_2C_2 + \\
& (A-K_1C_1)^TP_{23}(t+1)K_2C_2 + L_1^TR_1L_1,
\end{aligned}
\tag{6.32a}
$$

$$
\begin{aligned}
(P_{12}^T+P_{23})_t = {} & L_1^TB_1^T(P_{11}+P_{13})_{t+1}(A+B_2L_2) + \\
& (A+B_1L_1-K_1C_1)^T(P_{12}^T+P_{23})_{t+1}(A+B_2L_2) + \\
& L_1^TB_1^T(P_{12}+P_{22})_{t+1}K_1C_1 + (A-K_1C_1)^TP_{22}(t+1)K_1C_1
\end{aligned}
\tag{6.32b}
$$

Similar expressions hold for $P_{13}^T + P_{23}^T$ and $P_{13}^T + P_{33}$ and for the blocks of Σ.

Only for the trivial case $B_1 = 0$, $B_2 = 0$, $C_1 = 0$, $C_2 = 0$, the severe restriction (6.31) could be found to be fulfilled. This implies that $P_{ij} = 0$, $\Sigma_{ij} = 0$ for all blocks of P and Σ except P_{11} and Σ_{11}.

In all other cases the equat'ons for K_i and L_i do not separate. A fortiori this holds for the Stochastic Nash Compensator. By the next example we will corroborate this conclusion. In case of a two-stage control problem ($t_f = 2$), we will display the explicit coupling between the control and the filter gain.

Example 6.7

In this section we will solve the Stochastic Team Compensator problem for the case $t_f = 2$. We will consider the model of section 6.3.3, and make one simplifying assumption, viz. $x(0) \in G(0,\Sigma)$. Denote $Q_f = Q_2$.

From Lemma 6.5 and Proposition 6.6 we infer the following problem.

Problem. Find $K_i(t)$, $L_i(t)$, $i = 1,2$, $t = 0,1$ such that the following equations hold:

the state equation, $\Sigma(t+1) = \tilde{A}(t)\Sigma(t)\tilde{A}^T(t) + \tilde{M}(t)V\tilde{M}^T(t)$,

$\Sigma(0) = \mathrm{diag}(\Sigma,0,0)$,

the costate equation, $P(t) = \tilde{A}^T(t)P(t+1)\tilde{A}(t) + \tilde{Q}(t)$,

$P(t_f) = \mathrm{diag}(Q_2,0,0)$,

the first-order conditions,

$$B_1^T(E_1+E_2)^T P(t+1)\tilde{A}(t)\Sigma(t)E_2 + R_1L_1(t)E_2^T\Sigma(t)E_2=0,$$

$$B_2^T(E_1+E_3)^T P(t+1)\tilde{A}(t)\Sigma(t)E_3 + R_2L_2(t)E_3^T\Sigma(t)E_3=0,$$

$$E_2^T P(t+1)\tilde{A}(t)\Sigma(t)(E_1-E_2)C_1^T+E_2^T P(t+1)E_2K_1(t)N_1VN_1^T=0,$$

$$E_3^T P(t+1)\tilde{A}(t)\Sigma(t)(E_1-E_3)C_2^T+E_3^T P(t+1)E_3K_2(t)N_2VN_2^T=0$$

for $t = 0$ and $t = 1$.

From the first-order conditions we obtain eight equations in eight unknown, i.e. $\{L_i(t), K_i(t), i = 1,2, t = 0,1\}$. These eight equations can be evaluated using the state and costate equation. In particular we calculate the product $P(t+1)\tilde{A}(t)\Sigma(t)$ for $t = 0,1$; due to the choice of $\Sigma(0)$ and $P(t_f)$, the second and third row of block matrices in

$P(t+1)\tilde{A}(t)\Sigma(t)$ are zero ($t = 0,1$). Hence, four first-order conditions remain which can be written as:

$$L_1^T(1)[B_1^TQ_2A^2\Sigma C_1^T + (B_1^TQ_2B_1+R_1)L_1(1)K_1(0)(C_1\Sigma C_1^T+N_1VN_1^T) + B_1^TQ_2B_2L_2(1)K_2(0)C_2\Sigma C_1^T] = 0 \tag{6.33a}$$

$$L_2^T(1)[B_2^TQ_2A^2\Sigma C_2^T + (B_2^TQ_2B_2+R_2)L_2(1)K_2(0)(C_2\Sigma C_2^T+N_2VN_2^T) + B_2^TQ_2B_1L_1(1)K_1(0)C_1\Sigma C_2^T] = 0 \tag{6.33b}$$

$$[B_1^TQ_2A^2\Sigma C_1^T + (B_1^TQ_2B_1+R_1)L_1(1)K_1(0)(C_1\Sigma C_1^T+N_1VN_1^T) + B_1^TQ_2B_2L_2(1)K_2(0)C_2\Sigma C_1^T]K_1^T(0) = 0 \tag{6.34a}$$

$$[B_2^TQ_2A^2\Sigma C_2^T + (B_2^TQ_2B_2+R_2)L_2(1)K_2(0)(C_2\Sigma C_2^T+N_2VN_2^T) + B_2^TQ_2B_1L_1(1)K_1(0)C_1\Sigma C_2^T]K_2^T(0) = 0 \tag{6.34b}$$

If (6.33a) is postmultiplied by $K_1^T(0)$ and (6.34a) is premultiplied by $L_1^T(1)$, two identical equations emerge (similarly for (6.33b) and (6.34b)). This fact suggests that only the two products $L_1(1)K_1(0)$ and $L_2(1)K_2(0)$ can be determined from (6.33) and (6.34).

Indeed, these two products are all that are required to specify the control law. Since $z_i(0) = E[x(0)] = 0$ we have from (6.25) and (6.26)

$$u_i(0) = 0, \; i = 1,2$$
$$u_i(1) = L_i(1)K_i(0)y_i(0), \; i = 1,2.$$

It can be shown that $E[J(u_1,u_2)]$ for these values of $u_i(t)$, $t = 0,1$, only depends on $L_1(1)K_1(0)$ and $L_2(1)K_2(0)$, beyond known matrices.

□

6.3.4. Conclusions on the compensator approach

Stochastic control problems have been considered from the following point of view. The control strategy is supposed to be generated by a linear system (called a compensator). The stochastic control problem can be transformed into a deterministic optimization problem in some parameter space. As an example we solved the linear-quadratic Gaussian control problem for one decision maker. This result shows the potential of the method; it can be obtained, however, only by a suitable choice of the structure of the compensator and by imposing the appropriate properties on it.

The same procedure is followed to solve control problems with two decision makers, again in a LQG-framework. Known results for the global dynamics, shared information case could be recovered via the compensator approach. In the global dynamics, non-shared information case the following situation results. Each decision maker must manipulate two time-varying gain parameters, to be interpreted as the filter gain and the control gain. Two recursions can be recognized: a forward recursion for $\Sigma(t)$ (the covariance of the augmented system, consisting of the compensators' states and the state of the original system), and a backward recursion in $P(t)$ ($P(t)$ represents the costs incurred due to the controlling of the augmented system). By analogy to the case with one decision maker, one could expect that the control gains only depend on $P(t)$ and the filter gains only on $\Sigma(t)$. Conditions under which such a separation holds could not be established. Hence, in the determination of the optimal filter and control gains a coupling between $\Sigma(t)$ and $P(t)$ has been established. This coupling is to be interpreted as the interaction between control and information. The desirable property of the solution of the LQG-problem, as discussed in chapter 5, is precisely that the problem of control and filtering is separated (see Appendix 5D).

From the analysis of the Stochastic Team Compensator problem (section 6.3.3) it may be conjectured that under some conditions on the system parameters, the shared information case (i.e. the observation vector $y = (y_1;\ y_2)$ is shared by the agents) arises as a special case. However, this is prohibited by the choice of the structure for the compensator. In the shared information case, the structure of the compensator could be taken as follows. Rewrite the model as

$$x(t+1) = Ax(t) + (B_1,\ B_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + Mv(t)$$

$$y(t) = Cx(t) + Du(t) + Nv(t),$$

then the structure of the compensator equals the structure of the Kalman filter

$$\hat{x}(t+1) = A\hat{x}(t) + (B_1, B_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + K(t)[y(t)-Du(t)-C\hat{x}(t)] \quad (6.35)$$

$$u_1(t) = L_1\hat{x}(t)$$

$$u_2(t) = L_2\hat{x}(t)$$

(6.35) is the Kalman filter (cf. 5.43) and K(t) is given by (5.28). $(L_1(t), L_2(t), t \in T)$ follow by application of the minimum principle, as in Proposition 6.4. The compensator structure in (6.35) will be confronted with the set-up in section 6.3.3, in particular (6.25) and (6.26). In (6.25) and (6.26), the terms $B_2u_2(t)$ and $B_1u_1(t)$ respectively, are missing, compared to (6.35). This 'defect' must be compensated by the multiplicative factor $K_i(t)$, $i = 1,2$ in (6.25) and (6.26). Apparently this destroys the separation result, as established in the shared information case.

The ultimate conclusion is that the structure of the compensators (6.25) and (6.26) for the Stochastic Team Compensator problem does not seem to allow analytic results. Essentially, there is a coupling between information and control; this indicates that the control not only must be used to steer the system in some optimal way, but also to improve the quality of the observations. For the Stochastic Team Compensator problem, the control and filter gains can only be computed via rather complicated numerical problems; this limits insight into the solution to some extent.

6.4. The decentralized state-feedback control problem

6.4.1. Introduction

In the previous section we considered a dynamic system (compensator) that generated the control strategy. A further simplification can be obtained by omitting the dynamics in the compensator. Hence set $F_i(t) = 0$, $G_i(t) = 0$ in (6.4), then it is possible to eliminate the compensator state $c_i(t)$ and there remains essentially a restriction of the form

$$u_1(t) = L_1(t)y_1(t)$$
$$u_2(t) = L_2(t)y_2(t) \tag{6.36}$$

The control $u_i(t)$ is a direct output feedback of the information $y_i(t)$, $i = 1,2$. Note that this is a special case of the assumption that $u_i(t)$ is a linear function of all past measurements (see section 6.2, where we proposed this assumption to avoid the second guessing problem).

An alternative interpretation to (6.36) will be given. Consider the deterministic state equation

$$x(t+1) = Ax(t) + Bu(t) \tag{6.37}$$

with $x = (x_1;\ x_2)$, $u = (u_1;\ u_2)$.
Under the standard, quadratic cost function the optimal feedback law is of the form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}_t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t \tag{6.38}$$

Now suppose that DMi controls $u_i(t)$ and has only access to $x_i(t)$. Hence, we impose a fixed structure on the control gain in (6.38), such that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}_t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t \tag{6.39}$$

The problem is to determine $(F_1(t), F_2(t), t \in T)$ such that the standard quadratic cost function is minimized. The structure of the control gain in (6.39) might arise due to communication or information constraints faced by the decision makers. This constraint can be translated into (6.36) by constructing output equations of the form

$$y_1(t) = x_1(t) = C_1x(t)$$
$$y_2(t) = x_2(t) = C_2x(t),$$

for appropriate matrices C_1 and C_2.

Then

$$u_1(t) = F_1x_1(t) = F_1y_1(t)$$
$$u_2(t) = F_1x_2(t) = F_1y_2(t)$$

Summarizing, the following problem will be studied here.
The state equation is

$$x(t+1) = Ax(t) + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} u(t) \tag{6.40}$$

$$x = (x_1;\ x_2);\ u = (u_1;\ u_2)$$

The cost function is

$$J(u) = \sum_{t=0}^{t_f-1} (x^TQx+u^TRu)_t + x^T(t_f)Q_fx(t_f) \tag{6.41}$$

$$Q = \mathrm{diag}(Q_1, Q_2),\ R = \mathrm{diag}(R_1, R_2),\ Q_f = \mathrm{diag}(Q_{1f}, Q_{2f})$$

The decentralized state-feedback control law satisfies

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}_t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t$$

The corresponding control problem is to determine $(F_1(t), F_2(t), t \in T)$ such that (6.41) is minimized, subject to (6.39) and (6.40).

The B, Q, R, Q_f-matrices are restricted to be block-diagonal, in view of the structure of (6.39). The fact that A is not a block-diagonal matrix represents the interaction between the states x_1 and x_2, corresponding to the decision makers. The assumption on B, Q, R, Q_f can be relaxed at the expense of cumbersome computations later on. The state equation (6.40) is chosen to be deterministic, for convenience of notation only. Gaussian system representations can be considered as well, but will be transformed into deterministic recursions anyhow (e.g. as in Lemma 6.1).

6.4.2. Decentralized state-feedback: solution by the minimum principle

The decentralized state-feedback control problem will be transformed into an optimization problem for the gains $(F_1(t),\ F_2(t),\ t \in T)$.

Lemma 6.8. The decentralized state-feedback control problem can be reformulated as

$$\underset{(F_1,\ F_2)}{\text{minimize}}\ J(F_1,\ F_2) = \sum_{t=0}^{t_f-1} \operatorname{trace}\{[Q+F^T(t)RF(t)]X(t)\} + \operatorname{trace}[Q_f X(t_f)]$$

subject to

$$X(t+1) = (A+BF(t))X(t)(A+BF(t))^T,\ X(0),$$

where

$$X(t) := x(t)x^T(t)$$
$$F(t) := \operatorname{diag}\{F_1(t),\ F_2(t)\}$$

Proof. Substitute (6.38) into (6.40) and (6.41), and use the definition for $X(t)$. The recursion in $X(t)$ follows, while $J(u)$ is replaced by $J(F_1,\ F_2)$.

□

Define $P : T \to R^{n\times n}$; $X : T \to R^{n\times n}$ as in Lemma 6.8.

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$, $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, conformably with the partitioning of x into $(x_1;\ x_2)$.

Application of the minimum principle to the decentralized state-feedback control problem yields the following proposition.

Proposition 6.9. First-order conditions for the minimization problem posed in Lemma 6.8 are

$$[B_1^T P_{11}(t+1)A_{11}+B_1^T P_{12}(t+1)A_{21}]X_{11}(t) + [B_1^T P_{11}(t+1)B_1+R_1]F_1(t)X_{11}(t) +$$
$$[B_1^T P_{11}(t+1)A_{12}+B_1^T P_{12}(t+1)A_{22}+B_1^T P_{12}(t+1)B_2F_2(t)]X_{12}^T(t) = 0 \qquad (6.42a)$$

and

$$[B_2^T P_{22}(t+1)A_{22}+B_2^T P_{12}(t+1)A_{12}]X_{22}(t) + [B_2^T P_{22}(t+1)B_2+R_2]F_2(t)X_{22}(t) +$$
$$[B_2^T P_{22}(t+1)A_{21}+B_2^T P_{12}^T(t+1)A_{11}+B_2^T P_{12}^T(t+1)B_1F_1(t)]X_{12}(t) = 0 \qquad (6.42b)$$

where

$$P(t) = (A+BF(t))^T P(t+1)(A+BF(t)) + F^T(t)RF(t) + Q$$

$$P(t_f) = Q_f$$

$$F(t) = \mathrm{diag}(F_1(t),\ F_2(t))$$

$$X(t+1) = (A+BF(t))X(t)(A+BF(t))_t^T,\ X(0)$$

Proof. Define the Hamiltonian

$$H(X(t),\ P(t+1),\ F_1(t),\ F_2(t)) =$$
$$\mathrm{trace}[(Q+F^T RF)X(t) + (A+BF)X(t)(A+BF)^T P^T(t+1)]_t$$

and evaluate H using the decomposition of P, X and A.
The first-order conditions follow from ${}^{\partial H}/_{\partial F_i(t)} = 0$, $i = 1,2$, and the recursions for X(t) and P(t) follow from conditions (6A.4) and (6A.5) respectively (see Appendix 6A).

□

6.4.3. Decentralized state-feedback: solution by Lagrangean theory

The result as formulated in Proposition 6.9 seems at first sight rather untractable. Therefore we will attempt to solve the decentralized state-feedback problem by another technique. We will view it as a mathematical programming problem subject to a set of constraints.

The condition $F = \mathrm{diag}(F_1, F_2)$ can be interpreted as a constraint on the elements of F. By introduction of the vec-operator this constraint can be stated in a convenient form.
Let $S \in R^{m\times n}$, then $\mathrm{vec}(S) \equiv S^c$ denotes the mn-dimensional column vector consisting of the stacked columns of A

$$S^c := [S_{*1}; S_{*2}; \dots; S_{*n}]$$

The condition of $F = \mathrm{diag}(F_1, F_2)$ with $F_i \in R^{m_i\times n_i}$, $i = 1,2$, $m_1 + m_2 = m$, $n_1 + n_2 = n$, can be translated into the linear equation $DF^c = 0$ by choosing D in the following way

$$\begin{aligned}
D &= \mathrm{diag}(D^{(1)}, D^{(2)}),\ D^{(1)} \in R^{n_1m_2\times n_1m},\\
&\qquad\qquad\qquad\qquad D^{(2)} \in R^{n_2m_1\times n_2m},\\
D^{(1)} &= \mathrm{diag}([0,I],\dots,[0,I])\\
&\quad \text{where } 0 = 0_{m_2\times m_1},\ I = I_{m_2},\\
D^{(2)} &= \mathrm{diag}([I,0],\dots,[I,0]),\\
&\quad \text{where } I = I_{m_1},\ 0 = 0_{m_1\times m_2}.
\end{aligned}$$

The counterpart of Lemma 6.8 is

$$\underset{(F(t))}{\text{minimize}}\ J(F) = \sum_{t=0}^{t_f-1} \mathrm{trace}[(Q+F^TRF)_tX(t)] + \mathrm{trace}[Q_fX(t_f)]$$

subject to the set of constraints

$$\begin{aligned}
&X_0 \text{ is given}\\
&X(t) - (A+BF)X(t-1)(A+BF)^T_{t-1} = 0,\ t = 1,\dots,t_f\\
&DF^c(t) = 0,\ t = 0,1,\dots,t_f-1
\end{aligned}$$

We will apply the results from Lagrangean theory to this constrained minimization problem.
The Lagrangean L is defined as

$$L := \mathrm{trace}[Q_f X(t_f)] + \sum_{t=0}^{t_f-1} \mathrm{trace}[(Q+F^TRF)_t X(t)] +$$

$$\sum_{t=0}^{t_f-1} \{\mathrm{trace}\ P^T(t+1)[X(t+1)-(A+BF)X(t)(A+BF)_t^T] + \lambda^T(t)DF^c(t)\} \tag{6.43}$$

where $P : T \to R^{n\times n}$ and $\lambda : T \to R^{m_1n_2+m_2n_1}$ are the Lagrange multipliers.

Proposition 6.10. Given the Lagrangean (6.43) for the decentralized state-feedback control problem. The first-order conditions are

$$\begin{bmatrix} X(t) \otimes (B^TP(t+1)B+R) & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} F^c(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} -(X(t) \otimes B^TP(t+1))A^c \\ 0 \end{bmatrix} \tag{6.44}$$

where

$$P(t) = (A+BF)^TP(t+1)(A+BF)_t + (F^TRF)_t + Q$$
$$P(t_f) = Q_f \tag{6.45}$$

Proof. (6.44) follows from $\partial L/\partial F^c(t) = 0$ and $\partial L/\partial\lambda(t) = 0$ and (6.45) from $\partial L/\partial X(t) = 0$.
These differentiations can be performed by expressing L in terms of $F^c(t)$ [use the Kronecker product $A \otimes B = [a_{ij}B]$ and the rules:

$$Y = AXB^T \leftrightarrow Y^c = (B \otimes A)X^c,$$
$$\mathrm{trace}\ Y^TAXB^T = Y^{cT}(B \otimes A)X^c]$$

□

Remark
From the construction of D in this section, one can deduce that $DD^T = I$. In (6.44), $\lambda(t)$ can be eliminated to yield

$$(I-D^TD)[(X(t) \otimes B^TP(t+1))A^c + X(t) \otimes (B^TP(t+1)B+R)F^c(t)] = 0$$

which is a linear equation in $F^c(t)$; in fact it is a complicated way to augment the two first-order conditions (6.42) given in Proposition 6.9.

The solution of (6.44) can be utilized to obtain additional information on the sensitivity with respect to changes in J due to changes in $F^c(t)$.

Conclusion

The decentralized state feedback control problem arises as a special case of the compensator approach. Again it is possible to rephrase the control problem as a (deterministic) optimization problem in some matrix-valued gain parameters. The gains can be determined via the numerical solution of a two-points-boundary-value problem. Two (essentially) equivalent techniques have been applied: the minimum principle and the Lagrangean theory for constrained minimization problems. Although the presentation of the results obtained by both methods differs they are essentially the same, and no new viewpoints can be discovered beyond the conclusions for the compensator approach (see section 6.3.4).

6.5. Two-points-boundary-value problems

From the Propositions 6.6, 6.9 and 6.10, a common feature can be recognized.
Let the unknown parameters be denoted as U(t), then the first-order condition is (in symbolic notations)

$$f_1(U(t), \Sigma(t), P(t+1)) = 0, \ t \in T \tag{6.46}$$

For $\Sigma(t)$ and $P(t+1)$ there exist a forward and a backward recursion, respectively

$$\Sigma(t+1) = f_2(\Sigma(t), U(t), P(t+1)), \ \Sigma(0) \tag{6.47}$$

and

$$P(t) = f_3(P(t+1), \Sigma(t), U(t)), \ P(t_f) \tag{6.48}$$

Now assume that, by fulfilling the conditions of an implicit function theorem, (6.46) can be restated as

$$U(t) = g(\Sigma(t), P(t+1)) \tag{6.49}$$

Then substitution into (6.47) and (6.48) yields a two-points-boundary-value problem in $\Sigma(t)$ and $P(t)$:

$$\begin{aligned} \Sigma(t+1) &= \bar{f}_2(\Sigma(t), P(t+1)), \ \Sigma(0) \\ P(t) &= \bar{f}_3(P(t+1), \Sigma(t)), \ P(t_f) \end{aligned} \tag{6.50}$$

For this type of problem standard numerical routines are available (Ortega and Rheinboldt, 1970).

An algorithm to solve (6.50) and (6.46) has the following structure

1. Find initial estimates for $U(t)$, $t \in T$
2. Obtain an explicit expression for $U(t)$, say (6.49)
3. Substitute the solution to problem (6.50) into (6.49) to update $U(t)$, $t \in T$.
4. Repeat the steps 2 and 3 until a desired degree of convergence is reached.

Appendix 6A. The discrete-time matrix minimum principle

Let $X : T \to R^{n_1 \times n_2}$ be the state,

$U : T \to R^{m_1 \times m_2}$ the control matrix,

$P : T \to R^{n_1 \times n_2}$ the costate,

$T = \{0,1,\ldots,t_f\}$ the time-index set

such that

(6A.1) $X(t+1) - X(t) = F(t, X(t), U(t)),\ X(0)$

(6A.2) $J(U) = \sum_{t=0}^{t_f-1} L(t, X(t), U(t)) + K(X(t_f))$

(6A.3) $H(X(t)), P(t+1), U(t)) := L(t, X(t), U(t)) + \operatorname{trace}[F(t, X(t), U(t))P^T(t+1)]$

where F(.), K(.), L(.) are appropriate, given functions.

Theorem 6A.1. If $U^*(t)$ is the optimal unconstrained control and ($X^*(t)$, $t \in T$) the corresponding state trajectory, then there exists a costate $P^*(t)$, $t \in T$ such that

(6A.4) $X^*(t+1) - X^*(t) = F(t, X^*(t), U^*(t))$

(6A.5) $P^*(t+1) - P^*(t) = - \left.\frac{\partial H}{\partial X(t)}\right|_*$

(6A.6) $X_0^* = X_0,\ P^*(t_f) = \left.\frac{\partial K(X(t_f))}{\partial X(t_f)}\right|_*$

(6A.7) $\left.\frac{\partial H}{\partial U(t)}\right|_* = 0_{m_1 \times m_2}$

Proof. Athans, 1968.

□

Appendix 6B. Differentiation rules and Kronecker calculus

1. Let $x \in R^n$, $y \in R^m$ then $\frac{\partial y}{\partial x} \in R^{m\times n}$

2. $A^c \equiv \text{vec}\, A = [A_{*1};\ldots;\, A_{*n}]$ for a matrix A with n columns.

3. $Y = AXB \leftrightarrow Y^c = (B \otimes A)X^c$

   $\text{trace}[AB] = (A^T)^{cT}B^c$

   $\text{trace}(Y^TAXB^T] = Y^{cT}(B \otimes A)X^c$

   $(A \otimes B)(C \otimes D) = AC \otimes BD$

   $\text{trace}[A \otimes B] = \text{trace}[A].\text{trace}[B]$

4. Let $f : R^{m\times n} \rightarrow R,\ y = f(X)$

   then

$$\frac{\partial y}{\partial X} = \left[\frac{\partial y}{\partial x}\right]_{ij} \quad i = 1,\ldots,m,\ j = 1,\ldots,n.$$

$$\frac{\partial\ \text{trace}[AX]}{\partial X} = A^T, \qquad \frac{\partial\ \text{trace}[AX^T]}{\partial X} = A$$

$$\frac{\partial\ \text{trace}[AXBX^T]}{\partial X} = A^TXB^T + AXB$$

$$\frac{\partial\ \text{trace}[AX^TB]}{\partial X} = BA, \ \frac{\partial\ \text{trace}[AXB]}{\partial X} = A^TB^T$$

CHAPTER SEVEN

THE LOCAL DYNAMICS, NON-SHARED INFORMATION CASE

7.1. Introduction

In section 4.2 we introduced a control system in which decision makers do not share their on- and off-line model data. We argued that such a situation could be modelled by a set of interconnected systems (or: local systems). Each decision maker is supposed to control his own subsystem based on observations of the state of his local system and based on information from the interaction input of other local systems. In this chapter we will present examples of applications that employ the concept of interconnected systems, and mention a few possible ways to analyse such systems.

No attempt will be made to present a comprehensive problem formulation for the local dynamics, non-shared information case. To some extent, a general theory is being developed at the moment in the literature on Large-scale Systems Theory (see Jamshidi, 1983; Singh and Titli, 1978; Sandell e.a. 1978; Drenick, 1981). The state-of-the-art within the field of large-scale systems is characterized by a large number of case studies for which results can be obtained due to a special structure imposed on the systems. In a similar perspective we will proceed and consider state estimation problems for specially structured elements of a set of local systems (the so-called decentralized state-estimation). In particular, the following assumptions will be made.

First, we assume that the on-line model data $\eta_t^{(i)}$ of DMi consist of two parts: the input-output data $\{y_i(s), u_i(s), s \leq t\}$ and an interaction input, denoted by $G_t^{(i)}$. The latter set represents the information flow from the other local systems. Assume that

$$\emptyset \neq G_t^{(i)} \underset{\neq}{\subset} \{y_j(s), u_j(s), s \leq t\}, \; i \neq j,$$

which refers to a situation 'between' the completely non-shared and the completely shared information case.

Secondly, we assume that each decision maker has obtained values for the model parameters by processing the on-line model data through some identification method. The decision makers are supposed not to share the off-line model data.

Thirdly, we will simplify the control problem considerably by assuming that the optimal control problem can be separated from the filter problem. As in chapter 6, there will be an interaction between control and information. The following procedure, known as enforced separation, will be applied: the filter problem will be solved independently of the control problem. Subsequently, an (optimal) control problem can be formulated in terms of the obtained state estimate. This approach must be seen as a first step towards combined filter and control problems for large-scale systems configurations.

The assumptions made above lead to a model representation, that will be analysed in the following sections. In the two-decision-maker case DMi is supposed to control and observe his local system S_i, $i = 1,2$. The generalization to the case with an arbitrary number of decision makers is straightforward for the type of models to be considered here.

The local system S_i is given by

$$
\begin{aligned}
S_i:\quad x_i(t+1) &= A_{ii}x_i(t) + B_iu_i(t) + h_i(x,u) + M_iv(t) \\
y_i(t) &= C_ix_i(t) + D_iu_i(t) + N_iv(t)
\end{aligned}
\tag{7.1}
$$

where $i = 1,2$, $x = (x_1;\ x_2)$, $u = (u_1;\ u_2)$, $\dim(x_i) = n_i$, h_i is a R^{n_i} valued function representing the interaction input from S_j, $i \neq j$.

It is assumed that the knowledge of $G_t^{(i)}$ admits constructing the function $h_i(x,u)$. h_i may be of the following forms

$$h_i(x,u) = B_ju_j(t),\ i \neq j \tag{7.2a}$$

or

$$h_i(x,u) = A_{ij}x_j(t),\ i \neq j \tag{7.2b}$$

Note that in the local dynamics, non-shared information case it is not meaningful to compose the states of the local systems into an overall state $x = (x_1;\ x_2)$ with input $(u_1;\ u_2)$ (unless a coordinator is present to whom all relevant information is transmitted). The major part of the literature on large-scale systems deals with cases in which this augmentation procedure can be performed; still exceptions can be found, some of which that will be mentioned in the next section.

A brief outline of this chapter follows. In section 7.2 we will mention a few economic applications of modelling local systems and approaches that have been followed to tackle the control problem. In section 7.3 we formulate and discuss a decentralized state estimation problem for the local system S_1 and S_2, given by (7.1). Consequences of information exchange between local systems will be examined in section 7.4. Conclusions will be stated in section 7.5.

7.2. Applications of local systems

A broad variety of questions can be posed for the subject of interconnected systems. In engineering applications the important question of stabilization has been studied extensively. Within the local system configuration each decision maker is supposed to receive observations from his local system and to affect it by means of his control inputs. The non-trivial question is how the overall system can be stabilized by the decision makers. The stability question is motivated by stability problems in electric power systems. The decentralized structure is imposed because a centralized observer would require excessive computational requirements and information gathering networks.

As we mentioned already in section 7.1 the theory for this subject is still being developed. Heuristic procedures for special problems have been suggested. Consider the local system (7.1) for which a number of possible approaches have been introduced.

First, suitable bounds for the interconnection term $h_i(x,u)$ can be developed such that the overall system can be stabilized. The decision makers are supposed to invoke local compensators, cf. chapter 6.

Secondly, the term $h_i(x,u)$ can be regarded as a disturbance term in (7.1); no statistical properties, except boundedness are assumed at the outset. Each decision maker resorts to a minimax control strategy, i.e. he minimizes his cost function subject to the most adverse (bounded) outcome of $h_i(x,u)$. This approach is feasible for a large class of problems with local dynamics; it is a drawback, however, that no attempt is made to infer (actively) knowledge from the other subsystems.

Thirdly, we mention the case in which two sorts of dynamics may be distinguished. Dynamics with slow modes (common knowledge to all decision makers) and dynamics with fast modes. The latter part has a weak coupling with the former part and belongs to the private information of the decision maker. This set-up allows for simplification in the model structure of the interconnected system. Through the application of design objectives (cf. section 3.4) a satisfactory dynamic response can be obtained for each subsystem (Khalil and Kokotovic, 1978).

Fourthly, for some kind of systems it is possible to impose a hierarchical or multilevel structure. Local subsystems are connected with a central coordinator who performs a global optimization problem. This topic will not be treated here; we refer to the extensive literature that exists on it.

Finally, we mention that informational exchanges are related to the notion of signalling. If the control action of DMi is transmitted to DMj, it is possible to transmit relevant information as well (if the channel has a large capacity). This subject leads to fairly complicated mathematics, as in Bismut (1973). The relevance for economic situations has been pointed out by Ho, Kastner and Wong (1978).

If a local system configuration is feasible for a physical system, the model inherits the special structure of the system. For economic systems this may be true as well. The major question in economics is not, as in engineering applications, one of stabilization but one of modelling. A few cases in which the local dynamics, non-shared information case is of interest will be discussed below.

First, we mention microeconomic applications, in particular oligopoly and duopoly situations and applications in the theory of finance. For example, in a duopoly two firms compete on the same market; their outputs and production functions are kept secret, but the prices they set on the market are common knowledge to both firms. Oligopoly models have been studied extensively within the context of game theory (Friedman, 1977; Koutsoyiannis, 1979, chapters 8-10). As another example we mention the stock market, in which economic agents want to predict future prices, e.g. stock prices or spot prices of oil or gold. Different agents may have different information. According to Fama (see Fama, 1970), information can be classified into three classes: the weak, the semi-strong and the strong information set. The weak set includes all past values of the process under consideration, the semi-strong set all publicly available past information, and the strong set all past information both public and internal to an economic agent. This means that a financial expert has all available information (a strong information set) and the layman can only know observed prices (a weak information set).

Secondly, we mention management applications, e.g. one firm or organization has a number of subdivisions. Here it may be realistic to impose the team concept, i.e. the divisions of the firm have a common objective, and they have different information (e.g. due to spatial constraints). The economic aspects of this approach have been treated by Marschak and Radner (1972). A hierarchical framework is very natural as well: actions of the divisions are coordinated at a supremal level. Note that in microeconomic applications usually the static case is considered.

Thirdly, we mention the theory of rational expectations (see Lucas and Sargent, 1982). One of the most vivid, recent discussions in macroeconomics is how to formulate the expectations of an economic agent and how to incorporate these expectations into an economic policy model. This discussion is relevant for policy evaluation by control techniques (the policy effectiveness debate, see Lucas (1976) and the references mentioned in Lucas and Sargent (1982)), and the formulation of game-theoretical models. Although the relation with game theory is still unexplored (the exception is Buiter, 1984), it seems that the arguments of section 4.2.1 are useful for such a discussion. Indeed, in the multi-agent-case the economic agents generally have different expectations with respect to the future development of the economy. Such a case may prove to be an impulse to study the local-dynamics case.

7.3. State estimation for interconnected systems

In chapter 6 we noted that the key problem in control problems for dynamic games in which agents have non-shared information, is the interaction between control and information. In the local dynamics case this is even more pertinent. In order to avoid this complication, we will simplify the problem and concentrate fully on the information problem. For the case of two local systems we will investigate how a decision maker can estimate his state based on information available to him. The information exchange from the other subsystem has been specified for each decision maker. The state estimator of the decision maker under this set-up is called a local filter by Sanders et al. (1974). We will present some results for the local filter problem below.

Consider model (7.1) for subsystem S_i, $i = 1,2$ and set $B_i = 0$, $D_i = 0$, following our assumption to ignore the control problem. Observations $y_i(t)$ are supposed to consist of two parts: observations of the state $x_i(t)$ and observations of the interaction input $h_i(x,u)$, for $i = 1,2$.

Under these assumptions the local systems S_i, $i = 1,2$, given in (7.1) become

$$S_1:\quad x_1(t+1) = A_{11}x_1(t) + h_1(x,u) + M_1v(t) \tag{7.3a}$$

$$y_1(t) = \begin{bmatrix} y_{11}(t) \\ y_{12}(t) \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1(t) \\ h_1(x,u) \end{bmatrix} + \begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix} v(t) \tag{7.3b}$$

$$S_2:\quad x_2(t+1) = A_{22}x_2(t) + h_2(x,u) + M_2v(t) \tag{7.4a}$$

$$y_2(t) = \begin{bmatrix} y_{22}(t) \\ y_{21}(t) \end{bmatrix} = \begin{bmatrix} C_{22} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_2(t) \\ h_2(x,u) \end{bmatrix} + \begin{bmatrix} N_{22} \\ N_{21} \end{bmatrix} v(t) \tag{7.4b}$$

where $v(t) \in G(0,V)$ is a white noise process and $\dim(x_i) = n_i$, $i = 1,2$.

The information patterns for DMi, $i = 1,2$ consist of

$$\begin{aligned} \eta_\sigma^{(1)} &= \{A_{11}, M_1, C_{11}, N_1, V\},\ N_1 := (N_{11};\ N_{12}) \\ \eta_\sigma^{(2)} &= \{A_{22}, M_2, C_{22}, N_2, V\},\ N_2 := (N_{22};\ N_{21}) \end{aligned} \tag{7.5}$$

$$\eta_t^{(1)} = \{y_1(s),\ s \leq t\},\ \eta_t^{(2)} = \{y_2(s),\ s \leq t\} \tag{7.6}$$

The decentralized state estimation problem consists of finding, for given off-line model data $\eta_\sigma^{(i)}$, the state estimator of $x_i(t)$ based on the on-line model data $\eta_t^{(i)}$, for $i = 1,2$.

Proposition 7.1

Given (7.3) and (7.4) with information patterns specified by (7.5) and (7.6), respectively. Assume that $N_{ii}VN_{ii}^T > 0$ for $i = 1,2$.

Consider the case of DM1. The unbiased, minimum-variance estimator of $x_1(t)$ is given by

$$\hat{x}_1(t+1) = A_{11}\hat{x}_1(t) + K_1(t)[y_{11}(t) - C_{11}\hat{x}_1(t)] + K_2(t)y_{12}(t)$$

where

$$K_2(t) = I$$

$$K_1(t) = [A_{11}\Sigma_1(t)C_{11}^T + (M_1-N_{12})VN_{11}^T][C_{11}\Sigma_1(t)C_{11}^T + N_{11}VN_{11}^T]^{-1}$$

$$\Sigma_1(t+1) = (A_{11}-K_1(t)C_{11})\Sigma_1(t)(A_{11}-K_1(t)C_{11})^T + (M-N_{12}-K_1(t)N_{11})V(M-N_{12}-K_1(t)N_{11})^T$$

The expression for $\hat{x}_2(t)$ in the case of the local system S_2 follows analogously by interchanging the indices.

Proof

Two arguments will be given.

I. Substitute $h_1(x,u) = y_{12}(t) - N_{12}v(t)$ into (7.3a), then

$$x_1(t+1) = A_{11}x_1(t) + y_{12}(t) + (M_1-N_{12})v(t).$$

Since $y_{12}(t)$ represents a known time-series, the Kalman filter which is known to be unbiased and has minimum error-covariance can be applied (Appendix 5B).

II. Define a compensator for S_1

$$c_1(t+1) = F_1(t)c_1(t) + G_1(t)y_1(t) \text{ for given } c_1(0)$$

Let $\dim(c_1) = \dim(x_1) = n_1$, and $c_1(0) = E[x(0)]$. $(F_1(t), G_1(t), t \in T)$ remain to be determined such that $c_1(t)$ is an unbiased, minimum-variance estimator of $x_1(t)$. Define $e_1(t) := x_1(t) - c_1(t)$, $G_1(t) = [G_{11}(t), G_{12}(t)]$, compatible with the partitioning of $y_1(t) = (y_{11}; y_{12})_t$.
Then

$$\begin{aligned} e_1(t+1) = {} & [A_{11}-F_1(t)-G_{11}(t)C_{11}]x_1(t) + F_1(t)e_1(t) + \\ & [I-G_{12}(t)]h_1(x,u) + (M_1-G_1(t)N_1)v(t) \end{aligned}$$

For unbiasedness we require:

$$\begin{aligned} & F_1(t) = A_{11} - G_{11}(t)C_{11} \\ & [I-G_{12}(t)]h_1(x,u) = 0 \end{aligned}$$

Because function h_1 may not be completely known, we set
$I - G_{12}(t) = 0$ for all $t \in T$.
Through the requirement of unbiasedness the error equation becomes

$$e_1(t+1) = [A_{11}-G_{11}(t)C_{11}]e_1(t) + [M_1-G_{11}(t)N_{11}-N_{12}]v(t) \qquad (7.7)$$

Define $\Sigma_1(t) = E[e_1(t)e_1^T(t)]$, then $(G_1(t), t \in T)$ follows from minimization of the final-time cost function

$$J := E[e_1^T(t_f)Q_f e_1(t_f)] = \mathrm{trace}[Q_f \Sigma_1(t_f)]$$

with respect to $G_{11}(t)$, subject to the covariance equation of (7.7). Q_f is an arbitrary symmetric positive-definite weighting matrix. Application of the matrix-minimum principle (see Appendix 6A) readily yields the expressions for $G_{11}(t)$ and $\Sigma_1(t)$, as given in the statement of the proposition.

□

Remark

The case of a linear interaction input term can be obtained immediately by setting $h_1(x,u) = A_{12}x_2(t)$ and $h_2(x,u) = A_{21}x_1(t)$. Attempts to state the linear interaction input case in a slightly more general format deserve special care. For example, consider the local system for DM1 of the form

$$x_1(t+1) = A_{11}x_1(t) + A_{12}x_2(t) + M_1v(t) \tag{7.8a}$$

$$y_1(t) = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix}_t = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t + \begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix} v(t) \tag{7.8b}$$

Following the steps of the second proof of Proposition 7.1, the unbiasedness condition implies the equation

$$A_{12} = G_{12}(t)C_{12}$$

to be solved for $G_{12}(t)$. Additional assumptions are required for a unique solution of $G_{12}(t)$. Inspection of (7.8) shows that A_{12} and C_{12} must be of such a form that the components of $x_2(t)$ which affect $x_1(t+1)$ in (7.8a), are also observed via $y_{12}(t)$ in (7.8b).

7.4. Information exchange between local systems

7.4.1. Introduction

We will consider the effect of information exchange between two local systems S_1 and S_2, as stated below.

$$S_1: \quad \begin{aligned} x_1(t+1) &= A_{11}x_1(t) + A_{12}x_2(t) + M_1v(t) \\ y_{11}(t) &= C_{11}x_1(t) + N_{11}v(t) \end{aligned} \tag{7.9a}$$

$$S_2: \quad \begin{aligned} x_2(t+1) &= A_{22}x_2(t) + A_{21}x_1(t) + M_2v(t) \\ y_{22}(t) &= C_{22}x_2(t) + N_{22}v(t) \end{aligned} \tag{7.9b}$$

In addition, information exchange between S_1 and S_2 will be specified. $(m_1(t), t \in T)$ is the on-line information exchange from S_2 to S_1, and $(m_2(t), t \in T)$ from S_1 to S_2. Also off-line model data may be exchanged. Several cases will be considered and compared.

The compensator approach will be used to investigate the effects of the various forms of information exchange. Since a compensator comprises the variables known to a decision maker we can state the compensators $c_1(t)$ and $c_2(t)$ of the decision makers as follows:

$$c_1(t+1) = F_1(t)c_1(t) + G_{11}(t)y_{11}(t) + G_{12}(t)m_1(t) \tag{7.10a}$$

$$c_2(t+1) = F_2(t)c_2(t) + G_{22}(t)y_{22}(t) + G_{21}(t)m_2(t) \tag{7.10b}$$

Let $c_i(0) = E[x_i(0)]$,

$\dim(c_i) = \dim(x_i)$, $i = 1,2$.

The following three cases will be considered, in sections 7.4.2, 7.4.3 and 7.4.4, respectively.

I. The case of local dynamics, in which

$m_1(t) = c_2(t)$

$m_2(t) = c_1(t)$, for all $t \in T$

II. The case of global dynamics, in which

$m_1(t) = 0$

$m_2(t) = 0$, for all $t \in T$

III. The case of global dynamics, in which

$m_1(t) = (y_{22}(t); c_2(t))$

$m_2(t) = (y_{11}(t); c_1(t))$, for all $t \in T$.

7.4.2. Local filters: the exchange of the compensator's state

Consider the local dynamics case in which the decision makers have local compensators of the forms

$$c_1(t+1) = F_1(t)c_1(t) + G_{11}(t)y_{11}(t) + G_{12}(t)c_2(t) \tag{7.11a}$$

$$c_2(t+1) = F_2(t)c_2(t) + G_{22}(t)y_{22}(t) + G_{21}(t)c_1(t) \tag{7.11b}$$

The information exchange between the subsystems is precisely the compensator's state.

Consider the error equation for DM1. Define $e_i = x_i - c_i$, $i = 1,2$. Then from (7.9a) and (7.11a)

$$\begin{aligned} e_1(t+1) = {} & (A_{11}-G_{11}(t)C_{11}-F_1(t))x_1(t) + (A_{12}-G_{12}(t))x_2(t) \\ & + F_1(t)e_1(t) + G_{12}(t)e_2(t) + (M_1-G_{11}(t)N_{11})v(t) \end{aligned}$$

For unbiasedness of both $c_1(t)$ and $c_2(t)$, we require

$$\begin{aligned} F_1(t) &= A_{11} - G_{11}(t)C_{11} \\ G_{12}(t) &= A_{12} \end{aligned}$$

The error equation for DM1 under the unbiasedness restriction satisfies

$$\begin{aligned} e_1(t+1) = {} & (A_{11}-G_{11}(t)C_{11})e_1(t) + A_{12}e_2(t) + \\ & (M_1-G_{11}(t)N_{11})v(t) \end{aligned}$$

A similar equation holds for DM2. The usual procedure to determine $G_{11}(t)$ and $G_{22}(t)$ in the error equations for DM1 and DM2 is to augment the error equations by $e := (e_1; e_2)$. The minimum-variance argument (cf. Proposition 7.1) can then be applied to determine $(G_{11}(t), G_{22}(t), t \in T)$.

Because we are considering the local dynamics case, this procedure of augmenting $e_1(t)$ and $e_2(t)$ into $e(t)$ is not feasible. Two suggestions will be made to resolve this problem. First, a coordinator who knowns the parameters of S_1 and S_2 can perform the global minimization problem for $(G_{11}(t), G_{22}(t), t \in T)$. Secondly, DM1 can determine $(G_{11}(t), t \in T)$ in a suboptimal way by setting $A_{12}e_2(t) = 0$ for all $t \in T$ and by applying the minimum-variance argument. Both suggestions require unsatisfactory additional assumptions.

7.4.3. Local filters: the global dynamics case without information exchange

Now we consider the global dynamics case. Both decision makers are supposed to know the parameters of the augmented model

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{t+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} v(t) \tag{7.12}$$

Let $x = (x_1;\ x_2)$.
The on-line model data can be assumed to be non-shared, due to $(m_i(t) = 0,\ t \in T)$, $i = 1,2$. The compensators (7.10) are modified into

$$c_1(t+1) = F_1(t)c_1(t) + G_{11}(t)y_{11}(t) \tag{7.13a}$$

$$c_2(t+1) = F_2(t)c_2(t) + G_{22}(t)y_{22}(t) \tag{7.13b}$$

Let $c_i(0) = E[x_i(0)]$
$\dim(c_i) = \dim(x)$, $i = 1,2$.
Note that the dimension of $c_i(t)$ in (7.13) has increased, compared to the compensators defined in section 7.4.1.

The problem is to find $(F_i(t),\ G_{ii}(t),\ t \in T)$ such that $c_i(t)$ is an unbiased, minimum-variance estimator of $x(t)$. This problem can be solved in a way analogous to the problem posed in section 7.3. The result is:

Proposition 7.2

Given the state representation (7.12) and compensators (7.13). The unbiased, minimum variance estimator of $x(t)$ for DM1 based on (7.13a) is given by

$$\begin{aligned} c_1(t+1) &= Ac_1(t) + G_{11}(t)[y_{11}(t) - C_1c_1(t)], \\ G_{11}(t) &= [A\Sigma_1(t)C_1^T + MVN_{11}^T][C_1\Sigma_1(t)C_1 + N_{11}VN_{11}^T]^{-1}, \\ \Sigma_1(t+1) &= A\Sigma_1(t)A^T + MVM^T - (A\Sigma_1(t)C_1^T + MVN_{11}^T). \\ &\quad (C_1\Sigma_1(t)C_1 + N_{11}VN_{11}^T)(A\Sigma_1(t)C_1^T + MVN_{11}^T)^T, \end{aligned}$$

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \begin{matrix} C_1 = (C_{11}, 0), \\ M = (M_1; M_2) \end{matrix}$$

Proof

Retrace the steps of the second proof of Proposition 7.1, using $y_{ii}(t) = (C_{ii}, 0)x(t) + N_{ii}v(t)$, $i = 1,2$ (see (7.9)) and assuming that $N_{ii}VN_{ii}^T > 0$, $i = 1,2$.

□

Note that DMi observes $(y_{ii}(t),\ t \in T)$, subject to the known model (7.12). Then (7.9) can be rephrased as (7.12) together with the observation equations

$$y_{11}(t) = (C_{11}, 0)x(t) + N_{11}v(t) \tag{7.14a}$$

$$y_{22}(t) = (0, C_{22})x(t) + N_{22}v(t) \tag{7.14b}$$

The decision maker DMi can estimate the state $(x_1; x_2)_t$ based upon his observation set $(y_{ii}(s),\ s \leq t)$, $i = 1,2$. This problem has been resolved in Appendix 5B (the Kalman filter). The application of Theorem 5B.2 immediately yields the result of Proposition 7.2.

The motivation for considering the global dynamics case with $(m_i(t) = 0,\ t \in T)$, $i = 1,2$ is the following. The result presented above will be compared to the result of Proposition 7.1, specialized for the linear interaction input case with $h_i = A_{ij}x_j$. For both cases we can compare the filter performance, i.e. the error covariance of the estimator. In the case of Proposition 7.1 the error covariance depends on N_{12}, to be interpreted as the accuracy of the interaction input measurement. In the global dynamics case the decision maker knows the off-line model data A_{22}, A_{21}, M_2, instead of the on-line model data $(y_{12}(t),\ t \in T)$. Hence we can assess how accurately $(y_{12}(t),\ t \in T)$ must be measured by means of N_{12}, such that the on-line information exchange leads to a better filter performance than the off-line information exchange. The result of this analysis has implications for the implementation of the filter. The exchange of off-line model data is of a different nature than the exchange of on-line model data. The former case relates to structural aspects, the latter case to the (reliable and on-going) transmission of variables.

7.4.4. The global dynamics case: exchange of the compensator's state and the observation vector

Consider the augmented system (7.12) for the combined observation equation $y(t) := (y_{11}(t);\ y_{22}(t))$, see (7.14). The state estimator $E[x(t)|F^y_{t-1}]$ for the global dynamics case is given by the Kalman filter (Appendix 5B).

Let us state the result for the Kalman filter in decomposed form

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_{t+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_t + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_t \left[\begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix}_t - \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_t \right]$$

A computation leads to the following expression for $\hat{x}_1(t)$

$$\hat{x}_1(t+1) = (A_{11}-K_1(t)C_{11})_t \hat{x}_1(t) + K_{11}(t)y_{11}(t) +$$
$$[K_{12}(t),\ A_{12}-K_{12}(t)C_{22}] \begin{bmatrix} y_{22}(t) \\ \hat{x}_2(t) \end{bmatrix}$$

This result can be seen as a compensator for DM1 with state $(\hat{x}_1(t), t \in T)$, observations $(y_{11}(t),\ t \in T)$ and information exchange $m_1(t) = (y_{22}(t),\ \hat{x}_2(t),\ t \in T)$. Now observe again the compensators c_1 and c_2 given by (7.10), for the global dynamics case with information exchange $m_i = (y_{jj};\ c_j)$, $i \neq j$. Apparently this information exchange is sufficient to allow the decision makers to state their Kalman filters. Hence, this case provides a lower bound for the filter performance, because the Kalman filter is the best linear state estimator.

7.4.5. Conclusion

For a set of interconnected systems without control inputs various sorts of interaction inputs have been specified. The three cases presented have been compared with respect to their filter performance, measured by the error covariance of the filter. One of the cases could be identified as the Kalman filter, thus providing a lower bound for the error covariance.

The other two cases are different in the following way. The exchange of on-line model data for a local-dynamics model is to be compared to the knowledge of shared off-line model data. A trade-off can be made in terms of the covariance $N_{12}VN_{12}^T$ of the noise of the interaction input. The comparison allows an assessment as to how accurate the interaction input must be to outperform structural information. The filter performance of these two cases may also be compared to the lower bound of the Kalman filter.

7.5. Concluding remarks on the local-dynamics, non-shared information case

In this chapter we have considered a class of interconnected systems in which the decision makers are supposed to control and observe their local systems and to receive additional information from the other systems. Application of and motivation for this type of models can be found in electric power networks and water resource systems. In the field of economics we mentioned applications in finance, duopoly and rational expectations models.

It is believed that the local dynamics, non-shared information case is relevant for economic theory and deserves considerable attention. The results of this chapter should only be considered as a first step towards a more general set-up which may ultimately lead to a definite approach in large-scale modelling of economic systems. Attention was restricted to a decentralized filtering problem from the perspective of enforced separation between control and filtering, and to the quality of information exchange. Some other, supplementary, topics will be mentioned below.

First, the notions of learning and adaptive control should take a central place; the information from the interaction input may be used to estimate adaptively the model parameters, or, in a more general context, to identify the model structure. Adaptive control problems for economic models with one agent have been discussed in Kendrick (1981).

The extension to the multi-agent case is virtually unexplored. Beyond severe fundamental problems the computational requirements will be even more prohibitive than in the single-agent-case.

Secondly, we note that all results in this book are based on the assumption that the agents use the same model set $\underline{M}$ (see definitions 3.2 and 3.3 in section 3.2.1). In terms of the definition for the Gaussian system representation this means that the agents have the same belief with respect to the Gaussian distribution of the noise. However, in the non-shared information case it may also be natural to assume that the agents have so-called different prior beliefs (see Harsanyi, 1968; Borkar and Varaiya, 1983; Tsitsiklis and Athans, 1984; Varaiya, 1984). In the latter three references the following question has been posed.

Suppose the agents make decisions according to their different prior beliefs about the state of the world, and suppose they exchange information on the decisions they made, would they end up with the same belief in the long run (i.e. asymptotic agreement)? The analysis of this kind of information-theoretic problems may provide a better understanding of the interaction between information and control.

CHAPTER EIGHT

POLICY EVALUATION FOR LINKED ECONOMETRIC MODELS

8.1. Introduction

In this chapter we will illustrate the use of stochastic dynamic games for policy evaluation. The global dynamics, shared information case for which algorithmic implementation is feasible, will be considered. We will adopt the LQG-setting, and pay attention to the practial consequences of the topics treated in chapter 5. Three issues will be distinguished: the transformation of the econometric model to a state-space representation of appropriate dimension; the analysis of the dynamic properties of the state-space model; the computation of the optimal control solutions for a stochastic dynamic game under the Nash and the Pareto concepts.

We will discuss these three issues for a macroeconometric model that has been estimated using observed economic data. We want to emphasize that preliminary steps must be taken, before the control approach can start. These steps include the transformation of an econometric model to state-space form. The state vector should be of an appropriately low dimension in order to prevent difficulties in computation. Furthermore, the economic interpretation of the control results requires an analysis of the dynamic properties of the state-space model. In this chapter we will report extensively on these investigations.

The econometric model used for the illustration of the global dynamics, shared information case, is the Interplay model. This model consists of a set of models for national economies, members of the Common Market. The governments are considered to be the decision makers in the game. In section 4.2.2, it has been argued that the global dynamics, shared information case may indeed be appropriate for this type of model. The observed data used for the estimation of the Interplay model are taken from the National Accounts of the several members under consideration.

Though the model has been estimated using real-world economic data, the results presented in this chapter must be appreciated as follows. The formulation of the conclusions based on model experiments are only examples of economic analysis. The results cannot be used to draw real-world conclusions for effective policy behaviour of the decision makers.

A summary of the contents of this chapter follows. The Interplay model will be introduced in section 8.2 and characterized in terms of a simple demand model for an open economy. We will state the set of target and instrument variables that the governments are supposed to choose. In sections 8.3 - 8.6 we will perform the investigations preliminary to the actual control experiments. The transformation of the econometric model to state-space form will be discussed in section 8.3. In section 8.4 the dynamic properties of the state-space model will be analysed. The theoretical results obtained in section 8.3 will serve as a guideline. We will set out the procedures by which the parameters of the control problem will be determined in section 8.5. The simulation results over the planning period (section 8.6) constitute the final preparation for the control experiment to be started in section 8.7. We will derive optimal solutions for the target and instrument variables, and a measure for the uncertainty by which the expected paths for the optimal target variables will be reached. A summary and conclusions are reported in section 8.8. The definitions of endogenous and exogenous variables, the model equations and the desired values for target and instrument variables are given in the appendices.

8.2. The econometric model

In this section we introduce the econometric model that will be used in this chapter. In its original form the Interplay model concerns six countries of the Common Market (Federal Republic of Germany, United Kingdom, Italy, France, Belgium and the Netherlands). In order to illustrate the theory, it suffices to consider the two-countries case. A model consisting of the Federal Republic of Germany and the Netherlands will be considered (the notations G and NL will be used in this chapter). We will first make some general remarks on the underlying economic theory, then we will describe the structure of the two submodels for G and NL and their linking block and, finally, we will present the target and instrument variables to be used by the decision makers.

8.2.1. An elementary macroeconomic demand model

As an introduction to the Interplay model, we present a simple (Keynesian) demand model for an open economy with government.

The equations of the model are

$$Z_t = C_t + I_t + G_t + X_t \tag{8.1a}$$

$$M_t = \mu\, Z_t + M_t^0 \tag{8.1b}$$

$$Y_t = Z_t - M_t \tag{8.1c}$$

$$C_t = \alpha\, Yd_t + C_t^0 \tag{8.1d}$$

$$Yd_t = Y_t - T_t \tag{8.1e}$$

$$I_t = I_t^0,\ G_t = G_t^0,\ X_t = X_t^0,\ T_t = T_t^0 \tag{8.1f}$$

Z_t stands for the total expenditures, C_t private consumption, I_t private investments, G_t autonomous expenditures, X_t exports, M_t imports, Y_t national income, Yd_t disposable income, T_t taxes. Index t denotes time, suffix 0 denotes an exogenous variable. μ and α are constant coefficients.

Equation (8.1a) defines Z_t, usually called the demand (or aggregate demand). (8.1b,d) are behaviourial equations for imports and consumption respectively. (8.1c) is an equilibrium equation and states that national production equals domestic expenditures. (8.1e) is a definitional equation for disposable income. G_t, I_t, X_t and T_t are taken to be exogenous in this model, by (8.1f). Model (8.1) can be used to investigate the behaviour of national income; for instance, variables Z_t, M_t, C_t and Yd_t may be eliminated to obtain an autoregressive equation in Y_t, depending on the exogenous variables. The response of Y_t, when the exogenous variables are disturbed by shocks, may be analysed by the policy maker, in order to judge the effectiveness of his policy.

The equations in (8.1) constitute the case of a submodel of the Interplay model. Demand Z_t will turn out to play an essential role. A model of a national economy is a refinement of (8.1): it incorporates also (un)employment, prices and wages. The symbols used and the equations for the submodels can be found in Appendix 8A and Appendix 8B respectively.

8.2.2. The model description of Interplay

A model for a national economy as used in the Interplay model consists of 11 behaviourial equations. The endogenous variables, which are explained by these behaviourial equations can be divided into five groups. First, two equations for the labour market, explaining unemployment and employment in the private sector. Secondly, private expenditures (private consumption and private investments). Thirdly, one equation for nominal wage per labourer and fourthly, four equations for the various prices in the model. These are the prices of private consumption, private investments, exports of goods and autonomous expenditures. Fifthly, two equations for imports and exports of goods.

A number of definitional equations completes the submodel. Demand Z_t (in the Interplay model denoted by e2) and national income Y_t (denoted by gvampp) have been defined similarly as in (8.1a) and (8.1c) respectively. In addition there are definitional equations for employment of labourers, total expenditures including stocks and net invisibles, disposable wage and non-wage income, wage costs per unit of product and price of total expenditures. A more elaborate discussion of and motivation for the structure of the model is given in Plasmans (1980).

Any pair of submodels in the Interplay model is linked by means of bilateral trade flows and corresponding bilateral prices. Since these prices cannot be observed directly for the complete sample period, they are derived by means of an economic model (Plasmans, 1984). The model for the bilateral trade flows is based on an import-allocation mechanism. The exports of goods of any country is allocated over the other countries and the rest-of-the-world. The imports of goods of country i from country j ($i \neq j$) constitute the behaviourial equations of the linking block. When two submodels of Interplay are being linked, the behaviourial equations for exports of goods of the submodels will be replaced by definitional equations provided by the linking block. Hence, in the version of Interplay consisting of linked models for NL and G, we will have 22 behaviourial equations (see Appendix 8B).

Some technical remarks on the construction of the submodels follow. By suitable assumptions and approximations (if necessary), every model equation is linear (or linearized) and has constant coefficients. The application of LQG-theory requires a linear model, although economic processes are believed to be nonlinear. Moreover, the structure of the economy is changing over time; this may be accounted for by the use of time-varying coefficients. The application of LQG-theory is not hampered if models with time-varying coefficients are used. However, no version of the Interplay model with time-varying coefficients is available.

Except for the unemployment and interest rates, all variables in the model are expressed as growth rates. It is assumed that the growth rate of the product (quotient) of two variables equals the sum (difference) of their growth rates. This so-called "first-order approximation" is satisfactory if the growth rates are small. However, from experiments with the model it turns out that occasionally this assumption is violated, among others for the exports and imports equations. This observation is relevant for the assessment of the simulation results (see section 8.4.1).

The behaviourial equations have been estimated by ordinary least-squares over the sample period 1953-1980. The coefficients of the definitional equations have been determined from data of the same period. Statistics and specification analysis for a version of Interplay based on the sample 1953-1975 have been reported in Plasmans (1980). For our purposes we only need the model equations and the standard deviations of the residuals, given in Appendix 8B.

8.2.3. Target and instrument variables for the Interplay model

The governments of the countries are considered to be the decision (policy) makers of the game. By using the instrument variables, they aim to achieve certain goals, expressed as target variables. The most important economic targets, as they are usually specified in macroeconomic policy models, are covered by the target variables of table 8.1.

Table 8.1. The target variables.

1. Change in unemployment
2. Change in price index of private consumption
3. Change in real disposable income per employee in the private sector
4. Change in gross value added per employee (including self-employed) in the private sector
5. Change in the share of country i in the imports of goods of country j

For convenience, we will use a concise terminology for these target variables: (change in) unemployment, inflation, purchasing power, labour productivity and market share respectively. In terms of the variables of the model (see Appendix 8A), the target variables are $\Delta\tilde{un}$, Pcp, Wd - Pcp - Emp, gvampp - Emps (in NL: e2 - Emps), $mg_{i,j} - mg_i$ $(i,j = NL, G, i \neq j)$.*)

The governments of NL and G are supposed to choose from the following instrument variables.

*) The Dutch target variable is $mg_{G,NL} - mg_G$, whereas the German target variable is $mg_{NL,G} - mg_{NL}$. An economic variable x of country i will be denoted by x_i or x(i), i = NL, G, when the context does not reveal to which country it refers.

Table 8.2. The instrument variables.

Description	Symbol
1. Government expenditures	eg
2. Wage transfers of households to government ("social security premiums") and direct wage taxes	TRhg + TDw
3. Transfers of government ("social security aid") and rest-of-the-world to households	TRgh + TRrh
4. Nominal wage bill of the government	Wg
5. Indirect taxes minus subsidies	TS (only in NL)
6. Exchange rate	MR
7. Long-term interest rate	$\widetilde{RL}$
8. Primary liquidities	L1 (only in NL)
9. Budget deficit of the government	Fg (only in G)

8.3. Transformation to state-space form

In this section we will deal with the practical considerations of the transformation of an economic model to state-space form. The theoretical results, derived in section 5.2, will be used. In addition we will employ the specific structure of the econometric model and improve the results that can be obtained by straightforward application of the propositions of section 5.2.

We will start the analysis by presenting a characterization of the Interplay model of Appendix 8B in terms of an ARX(p,q)-model, see (2.1). The model given in Appendix 8B consists of 46 endogenous and 15 instrument variables. Inspection of the lag structure of the model equations reveals an ARX(2,1)-model

$$y(t) = A_0y(t) + A_1y(t-1) + A_2y(t-2) + B_0u(t) + B_1u(t-1) + F_0d(t) + F_1d(t-1) + Mv(t) \tag{8.2}$$

The matrices A_0, A_1, A_2, B_0, B_1, F_0, F_1 can be found from the equations given in Appendix 8B, using the choice of the instrument variables of table 8.2. The matrix M follows from the estimates of the standard deviations of the residuals of the behaviourial equations (see Theil, 1971, p. 114). These estimates can be found in table 8.21 of Appendix 8B. No covariances between residuals of two different regression equations have been computed, hence M is a diagonal matrix. $(v(t),\ t \in T)$ is assumed to be white noise, with $v(t) \in G(0,I)$.

For the choice of the target variables made in table 8.1, it follows that a matrix H can be found such that

$$z(t) = Hy(t) \tag{8.3}$$

Hence the target variables z(t) do not depend explicitly on the instrument variables (cf. (2.2)).

The equations (8.2) and (8.3) can be transformed to state-space form, see section 5.2.1. Replace $F_0d(t) + F_1d(t-1)$ in (8.2) by the notation Fd(t), then the result is

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + \bar{F}d(t+1) + \bar{M}v(t+1) \\ y(t) &= Cx(t) + Du(t) \\ z(t) &= HCx(t) + HDu(t) \end{aligned} \tag{8.4}$$

The vector x(t) consists of (delayed) endogenous and (delayed) instrument variables. From Proposition 5.5, the state vector of (8.4) would have 46 + 3 + 4 = 53 components, since A_2 and B_1 contain 3 and 4 nonzero columns respectively. For purposes of computation a further reduction to smaller state vectors would be desirable. This issue will be discussed below.

The target variables z(t) can be constructed out of 16 endogenous variables (called the "target-endogenous variables"). Hence 46 - 16 = 30 "non-target" endogenous variables may, in principle, be eliminated from (8.2). This elimination can be done by substitution of non-target endogenous variables into the right-hand side of (8.2). Obviously, if an endogenous variable is autoregressive, it cannot be eliminated. The effect of such a substitution process is that the lag structure of the right-hand side of (8.2) may change: the degrees of the y- and u-polynomials may increase and a moving average process in v(t) may arise.

Suppose that such an effect indeed occurs, then a state vector can still be constructed, but it is very well possible that its dimension is greater than under direct application of Proposition 5.5. If so, the augmenting effect due to higher degrees of y- and u-polynomials is stronger then the reducing effect of elimination. The substitution process has an adverse effect on the state dimension, but this adverse effect need not be overly unfavourable, if the special structure of the noise v(t) and the sparseness of the matrices A_0, A_1, A_2, B_0, B_1 in (8.2) is taken into account. This will be accomplished in a two-step procedure.

First, the substitution procedure of non-target endogenous variables is applied, using the certainty equivalence result (see Appendix 5D). Under the certainty equivalence result, we replace Mv(t) by its expectation; the substitution process is applied to a deterministic model, corresponding to (8.2). A considerable reduction of the state dimension can be achieved. For the Interplay model, with the target variables given in table 8.1, the state dimension is reduced to 28 (according to Proposition 5.5: 53). However, in the control experiment, we can only compute the deterministic state trajectory $(x(t),\ t \in [t_0,t_f])$ and the optimal target path $(z(t),\ t \in [t_0,t_f])$ under the optimal control $(u^*(t),\ t \in [t_0,t_f])$. The covariances of the target variables are not available in this first step.

In the second step, we will apply the substitution process to (8.2), while the noise term Mv(t) is retained. Note that the noise term Mv(t) only affects the behaviourial equations of (8.2). Hence we suggest to take all non-target endogenous variables which occur in definitional equations as candidates for substitution. Then no moving average process in Mv(t) will arise. The result is a larger state vector than in the certainty equivalence approach (i.e. 35 vs 28). The advantage is that the covariances of the target variables can be computed (see Appendix 5D).

Control experiments will be run, following the procedure suggested above. First, we use the certainty equivalence result. For a relatively small state vector all control experiments are performed to obtain the optimal target path. Typically, a large number of runs is required to determine suitable values for the parameters of the control problem. Secondly, <u>one</u> control experiment is performed for a higher dimensional state-space model, affected by system noise. The ultimate values of the parameters for this experiment have already been determined in the first step. The expectation of the optimal target path to be found, is still the same as in the first step (due to certainty equivalence), but now we are able to compute the covariances of the target variables. By this two-stage procedure both the expectation and the covariances of the optimal path of the target variables are computed, while the computational effort has been minimized.

It should be emphasized that the substitution process displays certain intricacies. This will be illustrated by means of a simple example.

Example 8.3.

Consider the set of equations

$$y_1(t) = y_3(t-1) \tag{8.5a}$$
$$y_2(t) = y_3(t-2) \tag{8.5b}$$
$$y_3(t) = y_1(t) + y_2(t) + u(t) \tag{8.5c}$$

Substitution of (8.5a,b) into (8.5c) yields

$$y_3(t) = y_3(t-1) + y_3(t-2) + u(t),$$

for which a state-space representation exists with state $(y_3(t);\ y_3(t-1))$. However, elimination of $y_3(t)$ in (8.5) yields

$$y_1(t) = y_1(t-1) + y_2(t-1) + u(t-1)$$
$$y_2(t) = y_1(t-2) + y_2(t-2) + u(t-2)$$

The equations in $y_1(t)$ and $y_2(t)$ are both autoregressive, hence $y_1(t)$ and $y_2(t)$ cannot be eliminated. The maximum lag in the instrument variable has become 2. A state-space representation with state vector $(y_1(t);\ y_2(t);\ y_1(t-1);\ y_2(t-1);\ u(t-1))$ may be constructed.

From (8.5), we recognize the ARX(p,q)-model (2.1), with $k = 3$, $p = 2$, $q = 0$, $M = 0$.

From Proposition 5.1 we obtain a state dimension $n = pk + qm = 6$.

From Proposition 5.5 we obtain a state dimension n = 4. Indeed, from (8.5) we have:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The reduced form matrices are:

$$(I-A_0)^{-1}A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; (I-A_0)^{-1}A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; (I-A_0)^{-1}B_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Because $(I-A_0)^{-1}A_2$ has two zero columns, a four dimensional state vector with elements $\{y_1(t),\ y_2(t),\ y_3(t),\ y_2(t-1)\}$ arises.

□

From this example we draw the following conclusions. First, the substitution process of non-target endogenous variables admits a reduction in the state dimension. Secondly, there are also opposite effects, in the sense that the substitution process does not necessarily reduce the dimension of the state. There are two reasons: the lag structure of the model will change, i.e. in general the degrees of the lag-polynomials will increase, and the substitution process causes autoregressive endogenous variables which cannot be eliminated anymore. Thirdly, the order in which the elimination process takes place is crucial.

A heuristic elimination procedure, available as an interactive computer program, constructs a state-space representation for (8.2) with an acceptably low dimension. The user may choose the variables which are subject to elimination. By a suitable choice of these endogenous variables the reduction to state-space form in both cases discussed above can be performed.

8.4. Properties of the econometric model

The aim of this section is to explore the dynamic properties of the econometric model. The results of this analysis will be useful in various respects.

1. It will yield insight into the dynamic behaviour of the model. More specifically, questions of whether the model is suitable for the control approach or for the game approach may be investigated. The former question relates to the effective use of the instrument variables, the latter to the strength of the interaction between the decision makers.
2. In a control experiment many parameters must be determined. Partly these parameters follow from predictions of economic variables, partly they are determined by iterative procedures. A sensible choice of the parameters can only be made when knowledge of the properties of the model is available.
3. The results of this section are required for the interpretation of the control experiments in economic terms.

Three topics will be discussed below: the dynamic simulation, the stability of the model and the multipliers of target variables and instrument variables. The latter topics have been discussed at the theoretical level in section 5.3.

8.4.1. The dynamic simulation

The model (8.2) has been simulated over the sample period $[0,t_0]$ = [1953-1980]. The simulation is called dynamic if the observed values for the exogenous variables u and d are used to compute the endogenous variables $y(t)$ over $[0,t_0]$ by means of (8.2). The initial conditions for (8.2) are the observed values for $y(0)$, $y(-1)$, $y(-2)$. The noise term $Mv(t)$ is set at zero for all $t \in [0,t_0]$.

The dynamic simulation serves to compare the simulated values of $y(t)$ to the observed values of $y(t)$. If, in practical models, the simulated values do not track their observed values, other specifications for some of the behaviourial equations may be sought, or values of coefficients may be adjusted.

When a dynamic simulation over the sample period [1953-1980] is made, the Interplay model shows rather poor simulation results. Indeed, dynamic simulations over such a long period seem a rather severe test; however, simulations over shorter time periods, e.g. the last ten years, starting at 1970, did not appear to be quite satisfactory. The dynamic simulation path did not track the rather volatile observed behaviour nor the turning points. Because the model will be used only for methodological purposes, i.e. as an illustration of the control approach for stochastic dynamic games and not for real-world economic conclusions, we will not attempt to improve the simulation results systematically. Simulation results will be presented and analysed only for the planning period (see section 8.6).

Note that a more quantitative approach to the notion of goodness-of-fit is possible by means of performance indices (e.g. the ones introduced by Theil, see Maddala, 1979). Also statistical tests to check the whiteness of residuals (the observed minus the simulated values of the endogenous variables) may be invoked. Though these topics are believed to be of considerable importance, they will not be treated here.

8.4.2. Stability

The stability of the model will be investigated under the certainty equivalence result. Consequently, the noise term $Mv(t)$ in (8.2) will be ignored and (8.2) can be transformed to state-space form (8.4), with a 28-dimensional state vector and system matrix A. The eigenvalues λ of A, for which $|\lambda| > .001$, are given below

Table 8.4. Eigenvalues of system matrix A.

1	.766	7	.160
2-3	.413 ± .541 i	8-9	.057 ± .074 i
4	.505	10	-.079
5	.490	11	-.049
6	.299	12	.0012

Only twelve eigenvalues prove to be greater than .001 (in absolute value). This fact indicates that essentially only 12 equations of the model contribute to the dynamic behaviour of the model. The remaining equations describe static relations between the endogenous variables. This is somewhat surprising, since there are 22 behaviourial equations.

Apparently the model is asymptotically stable, since $|\lambda| < 1$ for all λ. The complex eigenvalues $.413 \pm .514\, i$ induce oscillating behaviour with a periodicity of $2\pi(\arctan(.514/.413)) = 7.03$ years. In economic terms, this means that the basic business cycle of the model is seven years. A cycle of this length could be retraced in the dynamic simulation over the sample and in the observed data. Economists, when using simple macroeconomic models for industrialized western economies, claim that the length of the postwar business cycle is approximately 6 years. An earlier version of the Interplay model, estimated from the sample 1953-1975, displayed a cycle of 7.5 years (Plasmans, 1981).

8.4.3. Multipliers

When an instrument variable is changed by a unit impulse, the quantitative effect in the target variables is displayed by the corresponding multipliers. A definition of the various types of multipliers has been given in section 5.3.5. It should be noted that the effect of an impulse at time t on a variable formulated as a growth rate is that the actual value of that variable attains a higher level at time t and remains at that level in consecutive years.

We will give two reasons for the importance of a multiplier analysis. First, a multiplier analysis should confront the results of the identification (model structure and parameter values) with the underlying economic theory. The signs of the multipliers between u and z must be in accordance with the direction of the effect of instruments on targets, as postulated by economic theory. In addition, the signs and magnitudes of the multipliers may be confronted with results obtained by other models. Secondly, the magnitude of the multipliers will indicate the effectiveness of economic policy making. In particular, the magnitude of the multipliers of German (Dutch) instrument variables and Dutch (German) target variables displays the strength of the interaction between the two submodels. Hence the results of the multiplier analysis will be useful for the interpretation of control results.

In table 8.5 we present the impact, the one-year delayed interim and the equilibrium multipliers of the target variables and the instrument variables of the Interplay model. Hence we display the instantaneous effect, the effect in the next year and the total effect (as the sum of effects in all consecutive years) upon a unit impulse of the instrument variables. For every instrument variable in tabel 8.5, column a denotes the impact multiplier, column b the one-year delayed interim multiplier and column c the equilibrium multiplier. All entries in table 8.5 must be multiplied by a factor .001.

Table 8.5. Multipliers (x 10^{-3}) of target and instrument variables of the Interplay model.

	Instrument variables								
	eg(G)			Wg(G)			TRgh+TRrh(G)		
Target variables	a	b	c	a	b	c	a	b	c
Purchasing power (G)	040	063	254	150	030	263	352	135	770
Labour product. (G)	097	021	179	063	008	104	154	050	317
Market share (G)	011	004	005	007	002	003	017	009	009
Unemployment (G)	-026	-017	-035	-017	-010	-020	-041	-032	-061
Inflation (G)	-048	-029	-071	014	-007	014	-102	-082	-183
Purchasing power (NL)	018	016	043	012	009	025	029	029	077
Labour product. (NL)	032	016	055	020	009	032	050	031	097
Market share (NL)	024	009	039	015	005	023	037	019	069
Unemployment (NL)	-003	-003	-009	-002	-002	-005	-005	-005	-015
Inflation (NL)	-013	-015	-037	-009	-009	-022	-021	-026	-066

	TRhg+TDw(G)			MR(G)			$\tilde{R}L(G)$		
	a	b	c	a	b	c	a	b	c
Purchasing power (G)	-391	-150	-855	035	048	172	001	010	063
Labour product. (G)	-171	-055	-352	085	-001	122	003	021	045
Market share (G)	-019	-010	-010	133	-022	104	032	1230	1273
Unemployment (G)	045	035	068	-022	-010	-024	-001	-006	-009
Inflation (G)	113	091	203	-042	-016	-048	-001	-011	-018
Purchasing power (NL)	-055	-034	-108	064	009	075	091	109	227
Labour product. (NL)	-032	-032	-086	037	019	060	052	083	180
Market share (NL)	-042	-021	-077	021	004	027	964	968	1936
Unemployment (NL)	005	005	017	-006	-003	-012	-008	-014	-036
Inflation (NL)	023	029	074	-027	-021	-052	-038	-069	-155

Table 8.5 continued

	Instrument variables								
	eg(NL)			Wg(NL)			TRgh+TRrh(NL)		
Target variables	a	b	c	a	b	c	a	b	c
Purchasing power (NL)	058	032	099	271	029	307	373	040	422
Labour product. (NL)	101	017	125	096	014	115	132	019	158
Market share (NL)	001	001	002	001	000	002	001	001	002
Unemployment (NL)	-009	-006	-020	-009	-005	-018	-012	-007	-025
Inflation (NL)	-042	-034	-085	-040	-031	-078	-055	-043	-108
Purchasing power (G)	003	001	007	003	001	007	004	002	010
Labour product. (G)	001	002	010	001	002	010	002	003	014
Market share (G)	035	008	047	033	007	044	046	009	060
Unemployment (G)	-001	-001	-002	-001	-001	-001	-001	-001	-002
Inflation (G)	-002	-001	-003	-001	-001	-003	-002	-002	-004
	TRhg+TDw(NL)			MR(NL)			$\tilde{R}$L(NL)		
	a	b	c	a	b	c	a	b	c
Purchasing power (NL)	-435	-046	-493	-002	-024	-044	-052	- 083	- 180
Labour product. (NL)	-154	-022	-184	-004	-041	-056	-091	- 109	- 227
Market share (NL)	-001	-001	-002	000	-000	-001	-964	- 968	-1936
Unemployment (NL)	014	008	029	000	004	009	008	014	036
Inflation (NL)	064	050	126	002	018	038	038	069	155
Purchasing power (G)	-002	-003	-016	000	-000	-004	-001	- 010	- 063
Labour product. (G)	-005	002	011	000	-001	-003	-003	- 021	- 045
Market share (G)	-053	-011	-070	-001	-014	-021	-032	-1230	-1273
Unemployment (G)	001	001	002	-000	000	001	001	006	009
Inflation (G)	002	002	004	-000	001	001	001	011	018

Table 8.5 (continued)

	Instrument variables								
	TS(NL)			L1(NL)			Fg(G)		
Target variables	a	b	c	a	b	c	a	b	c
Purchasing power (NL)	-042	-019	-070	000	067	110	000	000	000
Labour product. (NL)	-007	002	-027	000	116	139	000	000	001
Market share (NL)	-000	-000	-001	000	001	002	000	000	000
Unemployment (NL)	001	012	058	000	-010	-022	-000	-000	-000
Inflation (NL)	141	206	374	000	-048	-095	-000	-000	-000
Purchasing power (G)	-000	-000	-009	000	001	012	000	001	003
Labour product. (G)	-000	-001	-006	000	004	008	001	000	002
Market share (G)	-003	-006	-038	000	040	053	000	000	000
Unemployment (G)	000	000	001	000	-001	-002	-000	-000	-000
Inflation (G)	000	000	002	000	-002	-003	-001	000	-001

column a: impact multiplier

column b: one-year delayed interim multiplier

column c: equilibrium multiplier

Interpretation of the multipliers

Let us consider the effects of a positive shock in the instrument variables on the target variables. We observe that there are two types of target variables. The instrument variables eg, Wg, TRgh+TRrh in NL and G, MR, $\tilde{RL}$, Fg in G and L1 in NL all have a downward effect on inflation and unemployment, and an upward effect on purchasing power, labour productivity and market share. The effects are reversed, when we consider the remaining instrument variables (see table 8.2). This result agrees with the fact that the Interplay model is essentially a demand model. The interpretation of the economists is that inflation and unemployment will drop, and labour productivity, purchasing power and market share will rise, if the government stimulates demand. They can do this either through increasing the government expenditures, wage bill or social security aid, or through cutting down (indirect or direct) taxes and social security premiums. Note that the exception to this rule is Wg(G):

the impact and equilibrium multipliers of Wg(G) and Pcp(G) are slightly positive.

Some remarks concerning the multipliers for separate instrument variables follow. A government can stimulate demand directly through the expenditures eg, or indirectly through the private sector by means of increasing social security aid or wage bill. From the magnitude of the multipliers it can be seen that in most cases the indirect effect is greater. Note, however, that the variables are expressed as growth rates. If the government decides to stimulate the economy by injecting a fixed amount of money (either through eg, Wg, or TRgh+TRrh), the assessment that the indirect effect is greater may change. Apparently, the absolute values of the instrument variables are required to evaluate the effects of increasing government expenditures on various target variables.

We observe that an increase in governmental expenditures (eg) causes a decrease in the inflation rate (both in NL and G). This effect can be traced down as follows in the equations of the model (see Appendix 8B). An increase in eg causes an increase in the demand e2 (equations 13 en 29), hence the inflation rate will drop (equations 8 and 24). Additionally, an increase in eg causes an increase in the labour productivity (see table 8.5), hence wages w will increase (equations 10 and 26) and private investments (cp and ip) as well. Therefore there is a second upward effect on demand e2. From a calculation of the multipliers between e2 and eg it follows that the impact multiplier equals .179 in G and .118 in NL. These figures are greater than the coefficients of eg in the e2-equations (the coefficients are .104 for G and .071 for NL). The ratio between the impact multiplier between eg and e2 and the coefficient of eg in the e2-equation is called the "Keynesian multiplier". For G it is .179/.104 = 1.72; for NL it is .118/.071 = 1.66.

With respect to the effects of an increase in governmental expenditures (eg), we observe from table 8.5 that the inflation rate decreases. Usually an increase in eg implies a number of important effects on the economy: for instance, taxes or transfers to the government must be increased or the budget deficit must rise. The latter effect may cause an increase in inflation (or an increase in the long-term interest rate, or both). These effects, however, have not been modelled endogenously in the Interplay model.

The effects of taxes and other transfers to the government are similar to the instrument variables of expenditures and wage bill of the government (although the signs differ). Primarily they must be considered as instrument variables affecting demand.

Concerning the long-term interest rate ($\tilde{RL}$), we notice that only the difference ($\tilde{RL}_G$-$\tilde{RL}_{NL}$) matters. An increase of this difference stimulates bilateral imports of goods, see equations 35 and 36. This is especially profitable for NL, since imports and exports of goods are relatively more important for NL than for G. This can be seen from the model equations as follows. From equations 37 and 38, we conclude that Dutch imports depend more strongly on bilateral trade flows than German exports do; from equations 13 and 29, we conclude that the share of xg in demand e2 is larger in NL than in G.

Exchange rate MR has a small effect on the target variables of NL and a substantially greater effect on the ones of G. Note that also the signs differ. It can be shown that in G (NL) demand will rise (drop), when the exchange rate of G (NL) rises (see table 8.6). The situation in NL can be explained as follows. A rise in MR may be conceived as a devaluation of the Dutch guilder, which has an equally great effect on the price of imports of goods (Pmg), see equation 42. From equation 6, the rise of price Pmg is translated into a rise of the price of exports Pxg for 82%. When Pxg rises, there are two effects: first, Pe2 will rise (equation 16), hence e2 will fall (via equations 1 and 2). Secondly, the price of exports in dollars (Pxg\$) will drop, hence xg will increase (equation 38) and so will e2. Apparently, for NL the first effect turns out to be stronger than the second. Hence we may conclude that a devaluation of the Dutch guilder will not stimulate the demand in NL (at least, not in our hypothetical model). In G, a devaluation does stimulate demand. This result corroborates the common opinion that the Dutch economy is more "open" than the German economy. The multipliers of demand and exports of goods in G and NL, and exchange rate MR will clarify and complement the discussion above (see table 8.6). The entries in table 8.6 have the same meaning as the ones in table 8.5.

Table 8.6. Multipliers of demand, export and exchange rate.

Endogenous variable	Instrument variable			Endogenous variable	Instrument variable		
Demand	MR(NL)			Demand	MR(G)		
	a	b	c		a	b	c
e2(NL)	-004	-048	-072	e2(G)	156	026	200
e2(G)	001	-002	-004	e2(NL)	075	015	097
Export	a	b	c	Export	a	b	c
xg(NL)	165	-001	163	xg(G)	578	-132	400
xg(G)	002	-008	-008	xg(NL)	157	-007	152

column a: impact multiplier
column b: one-year delayed interim multiplier
column c: equilibrium multiplier

The effect of L1 in N1 is relatively small, compared to other instrument variables. It occurs only at one place in the model (equation 1); it may be suggested to combine it with L2. The signs of multipliers of L1 and L2 are the same.

Finally, we notice that the mutual effects on the two countries are rather small, especially the influence of NL on G. This will weaken the game-theoretical application; the difference between the cooperative and the competitive solution can be expected to be small.

8.5. The choice of parameters in the control experiment

The control problem under consideration requires the determination of a great number of parameters. In this section we will list all parameters and discuss how values should be attributed to them. The parameters are:

1. the ultimate choice of the instrument and target variables
2. the desired paths ($\bar{u}_i(t)$, $\bar{z}_i(t)$, $t \in [t_0,t_f]$, $i = 1,2$)
3. the length of the planning period $[t_0,t_f]$
4. the weighting parameters Q_i and R_i of the cost functions

5. the weighting parameter α for the Pareto concept
6. the uncertainty parameters MVM^T and $\Lambda(0)$, see Appendix 5D.

Ad 1. We will simplify the control experiment by limiting the number of target and instrument variables. From a mathematical point of view this restriction will not alter the experiment. From an economic point of view, the instrument variables Fg (in G), MR and $\tilde{R}L$ (in G and NL) and Ll (in NL) will be considered as uncontrollable exogenous variables, either because their effect is small, or because the decision maker is supposed to have little freedom in the manipulation of these instrument variables. The number of target variables will be limited too. We will omit the market shares, because they can only be manipulated effectively through the long-term interest rate $\tilde{R}L$, see table 8.5. This is particularly clear from equation 35, Appendix 8B, when we rewrite equation 35 as

$$mg_{G,NL} - mg_G = .075\ mg_G + \text{constant}$$

The constant appears, since $(\tilde{R}L_G - \tilde{R}L_{NL})$ is now assumed to be beyond the control of the governments. The Dutch target variable $mg_{G,NL} - mg_G$ can only be affected via $.075\ mg_G$.

Concluding, the control experiments will be performed with 4 (4) target variables for G (NL) and 4 (5) instrument variables for G (NL).

Ad 2. The desired instrument paths will be discussed. The decision makers are supposed to choose from two kinds of policy. For simplicity, two opposed cases have been taken, but a richer analysis is possible by considering more cases.

Policy I is a <u>restrictive</u> policy. Economists give the following description: governments attempt to cut down expenditures (eg, Wg, social security aid) in order to reduce the budget deficit. Governments seek less interference in the private sector, which is expressed by a moderate tax policy.

Policy II is a <u>stimulating</u> policy (in fact, it is less restrictive than policy I). In economic terms, the main objective of the government is to increase demand by higher values for eg, Wg, TRgh than under policy I.

An increase in the budget deficit is not strictly prohibited. Because these expenditures must be financed somehow, it is assumed that tax rates will be higher than under policy I.

The desired and anticipated paths for the exogenous variables can be found in Appendix 8C, table 8.23. For convenience, the desired instrument paths for policies I and II are also presented in table 8.7. If available, the observed values for the instrument variables have been used. The difference between desired instrument paths under policy I and policy II is taken constant over the planning period (the difference being 2%, except for eg). This set-up will facilitate the interpretation of the simulation results.

Table 8.7. Desired instrument paths.

Germany

	Instrument variables							
	eg(G)		Wg(G)		TRhg+TDw(G)		TRgh+TRrh(G)	
Year	Policy I	Policy II	Policy I	Policy II	Policy I	Policy II	Policy I	Policy II
1981	-1.84	-1.84	6.41	6.41	6.86	6.86	8.21	8.21
1982	-4.73	-4.73	2.76	2.76	5.66	5.66	6.26	6.26
1983	-3.0	-3.0	0.0	0.0	2.57	2.57	4.0	4.0
1984	-1.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1985	-1.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1986	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1987	-3.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1988	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0

Netherlands

	Instrument variables									
	eg(NL)		Wg(NL)		TRhg+TDw(NL)		TRgh+TRrh(NL)		TS(NL)	
Year	I	II	I	II	I	II	I	II	I	II
1981	-0.41	-0.41	2.13	2.13	4.18	4.18	10.86	10.86	2.75	2.75
1982	-3.25	-3.25	3.64	3.64	7.41	7.41	10.06	10.06	-0.70	-0.70
1983	1.70	1.70	0.69	0.69	8.83	8.83	5.31	5.31	3.61	3.61
1984	0.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1985	-1.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1986	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1987	-3.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1988	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0

The choice of the desired target paths will be discussed. These paths are based on policy plans of governments or of a planning staff, advising governments. We will assume that governments intend to restore purchasing power and labour productivity, to keep inflation low and to reduce unemployment. The actual values are subject to a certain degree of freedom: our choice is given in Appendix 8C, table 8.23.

Ad 3. The length of the planning period corresponds with a standard medium-term planning period of 5 years.

The most favourable circumstance for planning is that the sample period extends up to present time (say t_0) and the planning period starts at t_0. In the version of the Interplay model under consideration, the sample period ends at 1980. Therefore, it was decided to simulate the model until t_0 based on observed values for exogenous variables. Since most of the observed values for the exogenous variables are available up to and including 1983, it was decided to take x(1983) as the (simulated) initial state. The planning period then ends at 1988.

Ad 4. The weighting matrices Q_i and R_i appear in the cost function

$$J_i = E[\sum_{t=t_0}^{t_f} \{\|z_i(t)-\bar{z}_i(t)\|^2_{Q_i} + \|u_i(t)-\bar{u}_i(t)\|^2_{R_i}\}],\ i = 1,2 \quad (8.6)$$

An interpretation of the cost function J_i has been given in section 4.4. The trade-off between the attainment of the desired target and instrument paths is settled by the choice of Q_i and R_i. The following procedure determines experimentally the values for Q_i and R_i (see De Zeeuw, 1984, pp. 150-151).

First, we notice that no clear interpretation can be attached to the off-diagonal elements of Q_i and R_i. Hence, we will restrict Q_i and R_i to be diagonal matrices. Secondly, we fix the diagonal elements of R_i, say at unity. This is done for simplicity, since only the ratio between Q_i and R_i matters. Thirdly, the Q_i-matrices will be tuned in the following way. For large values of elements of Q_i, optimal controls and target paths are computed. This will indicate whether all desired target paths can indeed be reached by excessive use of the instrument variables. Then the values of the diagonal elements of Q_i will be decreased successively in a number of experiments. The ultimate aim is to achieve a satisfactory trade-off between the deviation from desired target paths and desired instrument paths. This trade-off may or may not occur. It is at the policy maker's discretion when he wants to stop.

Two remarks on how to increase the flexibility of the proposed procedure will be made. First, the control method will still hold if we set some of the diagonal elements of Q_i equal to zero. This is needed when the corresponding target variable "overshoots" its desired path too much, and when penalization is considered undesirable. Secondly, a differentiation of the diagonal elements of R_i may be used to express a different measure of flexibility in the manipulation of the instrument variables (cf. section 4.4, for the interpretation of the control term $\|u_i(t)-\bar{u}_i(t)\|^2_{R_i}$ in (8.6)).

Ad 5. We will discuss the choice of the weighting parameter α of the Pareto concept (definition 3.3). Parameter α, $0 < \alpha < 1$, reflects the relative strength of the decision makers. The decision makers must agree upon the value of α, for instance via a bargaining procedure. Formalized treatments of bargaining have been applied to an earlier version of the Interplay model, see De Zeeuw, 1984, ch. 3 and App. 6.2. We will only present the Pareto solution for the values $\alpha = .8$, $\alpha = .5$ and $\alpha = .2$.

Ad 6. The matrix M in (8.2) is restricted to be a diagonal matrix of dimension 22, corresponding to the 22 behaviourial equations of the Interplay model. The values of the diagonal elements are taken to be the standard deviations of the residuals of the behaviourial equations, which is a standard statistic in least-squares estimation. The noise v(t) is assumed to be white, with $v(t) \in G(0,I)$. This choice for M and v(t) reflects the estimation procedure and the assumptions by which the model has been estimated. Note that the reduced form of (8.2) will be used to construct the state-space model. Hence the diagonal matrix M is premultiplied by $(I-A_0)^{-1}$, which is an almost completely filled matrix.

$\Lambda(t_0)$, the initial covariance matrix of x(t) under the optimal control law is assumed to be diagonal. The diagonal elements consist of the sample covariances of the endogenous variables (the sample is 1953-1980). Only that part of x(t) that corresponds with y(t) and not the part that corresponds with delayed endogenous and exogenous variables, is used for the construction of $\Lambda(t_0)$.

The values for M and $\Lambda(t_0)$ are reported in Appendix 8B, table 8.21.

8.6. The reference simulation

The reference simulation arises if in a dynamic simulation of the model both decision makers use their desired instrument paths (table 8.7). Because the decision makers are supposed to use either policy I or policy II, four different policy combinations can be distinguished. The anticipated paths for the uncontrollable exogenous variables are the same in all four cases (see Appendix 8C, table 8.22).

Before presenting the results of the four combinations, the reason for computing the reference simulation paths will be explained. The reference simulation path z_{ref} exhibits the desired target paths $\bar{z}$ which are reached or partly reached, and the target variables which need additional control effort, so that $\bar{z}$ may be reached under that control. The decision maker would appreciate an overshoot of the desired target path; for instance, the downward drift of the inflation rate is more rapid than required by the desired target path. The quadratic cost function penalizes both negative and positive deviations from the desired target path equally. This feature, typical for the quadratic cost function, can be accounted for by a suitable choice of $\bar{z}_i$, $\bar{u}_i$, Q_i, R_i, $i = 1,2$. In our applications the target variables which do not require additional control effort are attributed a weight zero in the cost function. The desired paths ($\bar{z}_i(t)$, $\bar{u}_i(t)$, $t \in [t_0,t_f]$, $i = 1,2$) will be determined independently and are not subject to systematic change.

Reference simulation paths for combinations of policies I and II for NL and G are given below. Note that no control experiment has been performed yet. In section 2.1, the combination of z_{ref} and the desired paths for exogenous variables has been called a <u>scenario</u>. The scenarios for the 10 target variables of table 8.1 and the four combinations of policies I and II for NL and G are given in table 8.8.

Table 8.8. Reference simulation paths.

Netherlands: Policy I

Target variables	1981	82	83	84	85	86	87	88
Purchasing power (NL)	.65	.68	-1.88	.58	.85	-.17	-.27	-.17
Labour product. (NL)	-.70	.25	.23	1.93	1.45	-.13	.16	-.03
Market share (NL)	-2.74	-2.03	-1.65	-1.38	-1.53	-1.71	-1.58	-1.34
Unemployment (NL)	.56	.48	.42	.17	.04	.05	-.01	-.01
Inflation (NL)	6.93	6.00	5.62	4.92	4.35	5.09	5.71	6.13
Purchasing power (G)	1.52	1.08	.82	1.00	1.16	.55	-.47	-1.10
Labour product. (G)	1.65	2.19	2.03	2.54	1.86	.42	.81	1.13
Market share (G)	-1.78	-1.33	-1.10	.81	-.79	-.50	-.68	-.58
Unemployment (G)	.63	.59	.70	.43	.38	.54	.62	.62
Inflation (G)	4.99	4.36	4.41	3.69	3.52	3.74	3.81	3.89
Purchasing power (NL)	.65	.68	-1.82	.68	.94	-.07	-.14	-.03
Labour product. (NL)	-.70	.25	.33	2.06	1.56	.01	.34	.20
Market share (NL)	-2.74	-2.03	-1.57	-1.29	-1.46	-1.61	-1.46	-1.23
Unemployment (NL)	.56	.48	.41	.16	.03	.03	-.03	-.04
Inflation (NL)	6.93	6.00	5.58	4.84	4.26	5.00	5.60	6.01
Purchasing power (G)	1.52	1.08	1.52	1.54	1.68	1.12	.22	-.34
Labour product. (G)	1.65	2.19	2.33	2.83	2.13	.80	1.31	1.58
Market rate (G)	-1.78	-1.33	-1.06	.85	-.76	-.47	-.64	-.54
Unemployment (G)	.63	.59	.62	.32	.29	.44	.48	.49
Inflation (G)	4.99	4.56	4.21	3.53	3.46	3.66	3.66	3.76

Netherlands: Policy II

Target variables	1981	82	83	84	85	86	87	88
Purchasing power (NL)	.65	.68	-1.88	.92	1.25	.38	.37	.41
Labour product. (NL)	-.70	.25	.23	2.07	1.71	.27	.66	.42
Market share (NL)	-2.74	-2.03	-1.65	-1.37	-1.53	-1.70	-1.58	-1.34
Unemployment (NL)	.56	.48	.42	.16	.04	.07	.04	.05
Inflation (NL)	6.93	6.00	5.62	5.14	4.89	5.70	6.22	6.58
Purchasing power (G)	1.52	1.08	.82	1.01	1.17	.56	-.46	-1.09
Labour product. (G)	1.65	2.19	2.03	2.55	1.87	.43	.82	1.14
Market share (G)	-1.78	-1.33	-1.10	.85	-.71	-.40	-.56	-.49
Unemployment (G)	.63	.59	.70	.43	.38	.54	.62	.62
Inflation (G)	4.99	4.56	4.41	3.69	3.52	3.73	3.80	3.88
Purchasing power (NL)	.65	.68	-.82	1.01	1.34	.48	.50	.55
Labour product. (NL)	-.70	.25	.33	2.19	1.82	.40	.84	.59
Market share (NL)	-2.74	-2.03	-1.57	-1.29	-1.46	-1.61	-1.45	-1.23
Unemployment (NL)	.56	.48	.41	.15	.02	.05	.01	.02
Inflation (NL)	6.93	6.00	5.58	5.06	4.80	5.61	6.11	6.46
Purchasing power (G)	1.52	1.08	1.52	1.54	1.68	1.13	.23	-.33
Labour product. (G)	1.65	2.19	2.33	2.84	2.14	.81	1.33	1.59
Market share (G)	-1.78	-1.33	-1.06	.89	-.68	-.36	-.52	-.46
Unemployment (G)	.63	.59	.62	.32	.29	.43	.48	.49
Inflation (G)	4.99	4.56	4.21	3.53	3.45	3.65	3.66	3.75

Conclusions for the reference simulation

Fix the policy of NL, e.g. NL uses policy I. The use of policy II in G will lead in G to a higher rate of purchasing power, labour productivity and market share, and a lower rate of inflation and unemployment than the use of policy I. Over the period 1984-1988, the average difference for the five German target variables are .63, .37, .03, -.11 and -.13, respectively. The difference in the instrument variables between policy I and policy II is 2% (see table 8.7).

We conclude that the stimulating policy in G is better than the restrictive policy on all five target variables. The same conclusions hold if NL uses policy II.

Conversely, fix the policy of G, say policy I. The conclusion for NL is: the use of policy II yields a higher rate of purchasing power, labour productivity and market share than policy I. It also yields a higher rate of inflation and unemployment, contrary to the situation in G. The average differences for the five target variables of NL over the period 1984-1988 are .50, .17, 0.0, .02 and .24, respectively. Since the difference in the instrument variables between policy I and policy II is again 2%, we observe that the effects on purchasing power, labour productivity and market share of a shift from policy I to policy II are greater in G than in NL. If a shift from I to II occurs the changes in unemployment and inflation have different signs in NL and G. This is caused by the instrument variable TS(NL) which is absent in G. TS has (in absolute value) the greatest equilibrium multipliers for inflation and unemployment among all Dutch instrument variables (see table 8.5).

One of the four combinations of policies I and II has been selected for the control experiments. In figure 8.1 we present the reference simulation path z_{ref}, if G and NL follow policy II. The desired target paths are also shown in figure 8.1. The confrontation between z_{ref} and $\bar{z}$, for each of the target variables presented in figure 8.1, will be the basis for the optimal control experiment.

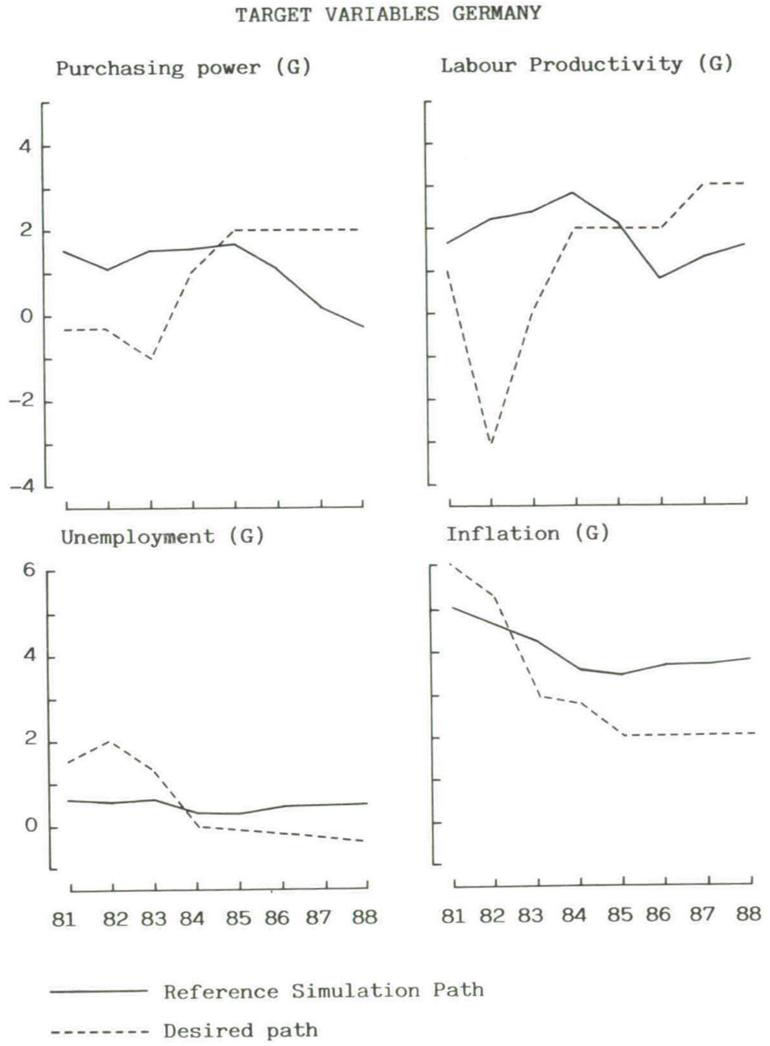


Figure 8.1. Reference simulation path and desired target path.

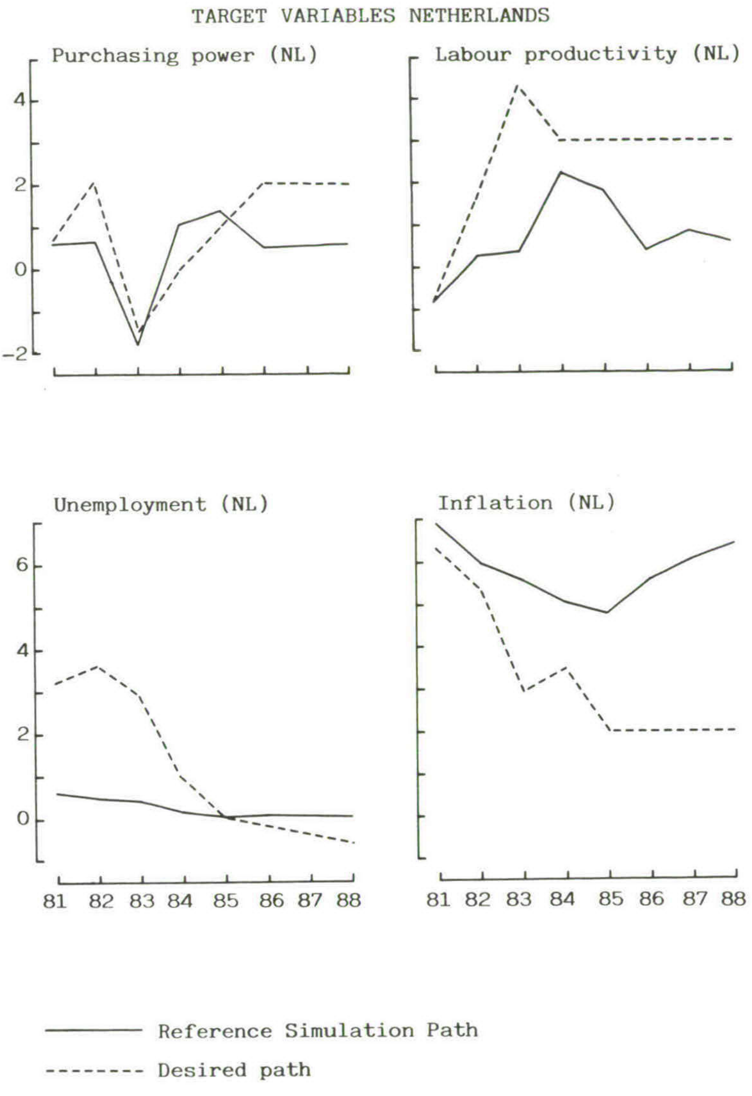


Figure 8.1 (continued) Reference simulation path and desired target path.

Digression: the need for parameter adjustments

When we consider figure 8.1, in particular the inflation rate in NL, we notice that in our example the simulated inflation rate in NL is unacceptably high, compared to the desired target path. It is known that the observed values for the inflation rate in NL in 1983 and 1984 are about 3% and are expected to remain at that level for the period 1985-1988. In this case the desired target path equals the anticipated path for the inflation rate. This discrepancy between the simulated and the anticipated path for inflation is caused by two properties of the hypothetical model. First, there is an intercept of 1.479 in the Pcp-equation (equation 8, Appendix 8B). This intercept causes an annual autonomous inflation rate of about 1.5%, which is realistic for the sample period, though not for the period 1983-1988. Secondly, consider wage equation 10 and rewrite .869 Pcp + .290 Pcp_{-1} as 1.159 $Pcp_{-\frac{1}{4}}$. Then equation 10 can be interpreted as follows: wages w are compensated for the rise in consumption prices by 116% with a delay of a quarter. Again, this is not realistic for the period 1983-1988. In both cases, the model specification based on the period 1953-1980 is not appropriate for predicting the inflation rate over the planning period.

More realistic predictions may be obtained if the model is adjusted. Possible suggestions are: either equation 10 is omitted from the model (hence wages w will become exogenous) or the term 1.159 $Pcp_{-\frac{1}{4}}$ is replaced by c $Pcp_{-\frac{1}{4}}$, where c is an appropriately chosen constant smaller than 1.

In order to show the feasibility of making adjustments, the former, though rigid, suggestion will be followed. Reference paths will be presented, taking wages w(NL) in the model (8.2) exogenously. Only the case that both NL and G use policy II will be considered. After having made the adjustment, the German target paths do not change significantly; hence, we will only present values of Dutch target variables. To show the impact of the adjustment, the simulated values of w(NL) in the original model and the Dutch target variables (see table 8.8) will be given too.

Table 8.9. Reference simulation for z_{NL} and w_{NL}.

A. w_{NL} as exogenous variable

Variable (NL)	1981	1982	1983	1984	1985	1986	1987	1988
wages w (exogenous)	4.11	6.18	3.59	2.5	2.5	2.5	2.5	2.5
purchasing power	.03	.46	-2.50	-.27	-.06	-.87	-.99	-1.09
labour productivity	-1.15	.24	-.14	1.43	1.08	-.29	.04	-.27
market share	-2.75	-2.02	-1.58	-1.29	-1.46	-1.61	-1.45	-1.22
unemployment	.60	.28	.35	-.05	-.39	-.45	-.50	-.55
inflation rate	5.58	5.87	4.06	2.64	2.32	3.15	3.28	3.39

B. w_{NL} as endogenous variable

Variable (NL)	1981	1982	1983	1984	1985	1986	1987	1988
wages w (endogenous)	6.49	6.51	6.20	6.84	6.96	6.84	7.45	7.88
purchasing power	.65	.68	-1.82	1.01	1.34	.48	.50	.55
labour productivity	-.70	.25	.33	2.19	1.82	.40	.84	.59
market share	-2.74	-2.03	-1.57	-1.29	-1.46	-1.61	-1.45	-1.23
unemployment	.56	.48	.41	.15	.02	.05	.01	.02
inflation rate	6.93	6.00	5.58	5.06	4.80	5.61	6.11	6.46

Discussion of the result

Before we discuss the specific outcomes, a general remark on the making of adjustments in econometric models will be made. Often in empirical models, the model performance is unsatisfactory; typical reasons are that economic time series cover long time periods, over which the structure of the economy changes, and that the data set does not contain enough information to keep statistical errors within bounds.

The model performance can be improved by making adjustments, e.g. by modifying a parameter value. Such adjustments require a clear understanding of the economic model and the economic reality, and must be motivated by a sound economic interpretation. The effect of adjustments may be considerable, and extensive experimenting should justify its use.

Now let us trace the effects of the adjustment made in our hypothetical example. In figure 8.2 below, the simulated target paths (except for the market share) are confronted with the desired target paths. After the adjustment, the inflation in NL is still somewhat higher than its desired value (approx. 1%). In figure 8.1 the difference was 4%. As an accompanying effect, unemployment in NL moves downwards, eventually to reach its desired path. The mechanism that causes the reduction in Pcp can be explained from the model equations, see Appendix 8B. On average, wages w are about 5% lower after the adjustment (see table 8.9). Hence Pcp will be about 3.5% lower through H($-\frac{1}{4}$), see equation 8 and 15. As an opposite effect, Pcp will increase, since cp and ip drop, and consequently e2 drops (see equations 1, 2, 13 and 8). The net result is shown in table 8.9: on the average Pcp (NL) drops about 3%. Because w (hence Wd, equation 11) and e2 drop, the remaining target variables Wd-Pcp-Emp and e2-Emps, now are considerably below their desired values (approx. 3%). The control effort must be shifted from inflation towards these target variables. This will become manifest, when we compute optimal controls for the adjusted model in section 8.7.4.

8.7. Control experiments: Nash and Pareto solutions

The optimal control solution derived in section 5.4 will be applied to the econometric model (8.2). The implementation of the optimal control solution is based upon the results obtained in the previous sections, i.e. the analysis of the properties of the model (section 8.4), the procedures to determine the parameters (section 8.5) and the reference simulation path (section 8.6). We will attempt to give an interpretation of the control solution in economic terms, although this interpretation may not be used for real-world conclusions.

A brief outline of the contents of this section follows. We will review the ultimate choice of the values of the parameters, then give the optimal paths for the target and instrument variables under the Nash and Pareto solution concepts, and discuss this result. Subsequently we will give the variances and correlation coefficients of the optimal target variables for the Nash solution, and repeat these results for

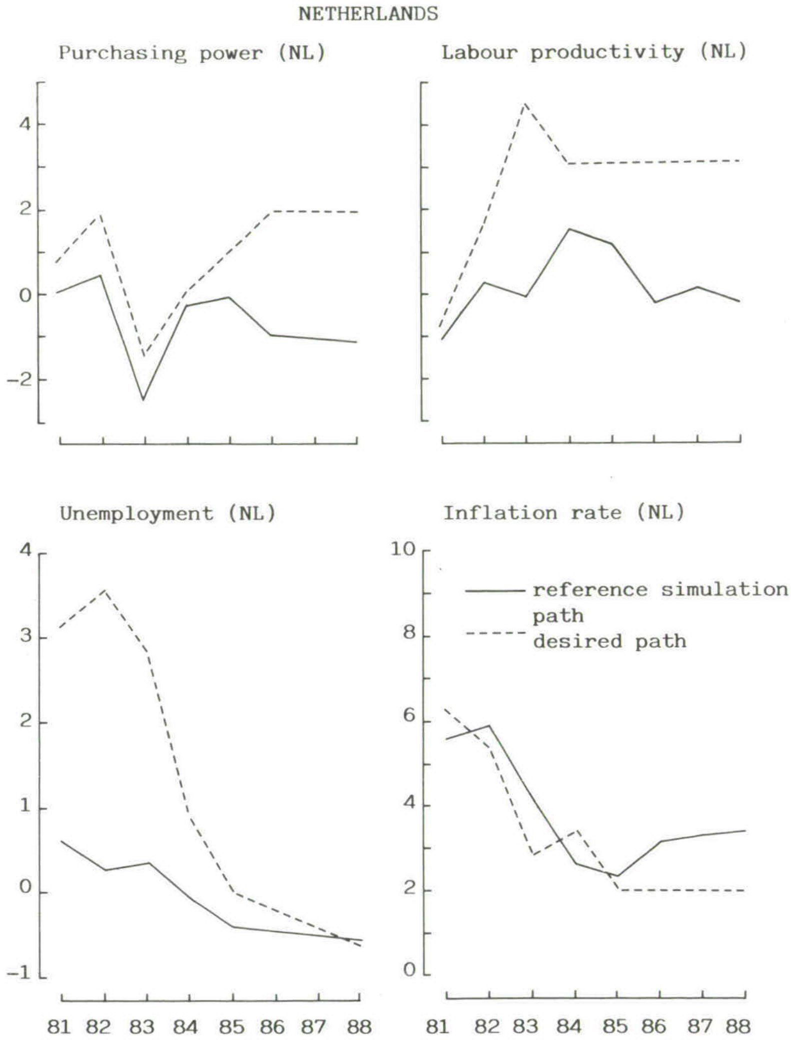


Figure 8.2. Reference Simulation Paths for Exogenous w(NL).

an adjusted model with exogenous Dutch wages w_{NL}. Finally, we will summarize and evaluate the control approach for linked econometric models as an example of economic reasoning in a methodological study.

8.7.1. The choice of the parameters

We will summarize the choice of the parameters for the optimal control problem. The parameter values are equal for the Nash and Pareto concepts, which makes comparison possible.

The target and instrument variables are

$$z_G = (\text{Wd-Pcp-Emp, gvampp-Emps}, \Delta\tilde{un}, \text{Pcp})_G$$

$$z_{NL} = (\text{Wd-Pcp-Emp, e2-Emps}, \Delta\tilde{un}, \text{Pcp})_{NL}$$

$$u_G \equiv u_1 = (\text{eg, Wg, TRhg + TDw, TRgh + TRrh})_G$$

$$u_{NL} \equiv u_2 = (\text{eg, Wg, TRhg + TDw, TRgh + TRrh, TS})_{NL}$$

The state vector in (8.4) has dimension n = 24, in the case of 8 target variables (using the certainty equivalence result). It consists of the following components

$$(\Delta\tilde{un}, \text{Emp}, w, cp_{-1}, \text{Pcp, Pip, Pxg, Pmg, Emps, gvampp},$$
$$\text{e2}, \text{e2}_{-1}, W_{-1}, H(-\tfrac{1}{4}), \text{Wd})_G \text{ and}$$
$$(\Delta\tilde{un}, \text{Emp}, w, \text{Pcp, Pip, Pe2, Emps, e2, Wd})_{NL}$$

The desired instrument paths for policy II for u_G and u_{NL} are

$$u_G(t) = (0, 0, 2, 4), \quad t = 1984,\ldots,1988$$

$$u_{NL}(t) = (0, 0, 2, 4, 2), \quad t = 1984,\ldots,1988$$

The desired paths for other exogenous variables and target variables are given in Appendix 8C, tables 8.22 and 8.23.

The weighting matrices Q_i and R_i for the cost function J_i in (8.6), i = 1,2, have been determined in a great number of experiments. The following results have been obtained, for a time-varying Q_G-matrix and other matrices constant.

$$\text{diag } (Q_G(t)) = (0,\ 10,\ 20,\ 20),\ t = 1983,\ldots,1986$$
$$= (0,\ 10,\ 30,\ 20),\ t = 1987,\ 1988$$
$$\text{diag } (Q_{NL}(t)) = (0,\ 10,\ 10,\ 20),\ t = 1983,\ldots,1988$$
$$\text{diag } (R_G) = (1,\ 1,\ 1,\ 1)$$
$$\text{diag } (R_{NL}) = (1,\ 1,\ 1,\ 1,\ 1)$$
$$Q_1 \equiv Q_G,\ Q_2 \equiv Q_{NL},\ R_1 \equiv R_G,\ R_2 \equiv R_{NL}.$$

The planning period is 1983-1988. The initial state vector x(1983) has been obtained by simulation of the model over 1980-1983, based on observed data for exogenous variables. The reference simulation paths have been given in figure 8.1, section 8.6.

The weighting parameter α for the Pareto concept is .5. This choice implies equal weighing of the cost functions J_1 and J_2. Additional information will be supplied for α = .8 and α = .2.

The values of the uncertainty parameters M and $\Lambda(t_0)$, see Appendix 5D, will be reported in Appendix 8B, table 8.21.

8.7.2. The Nash and Pareto solutions

For the choice of the parameters of the control problem as given in section 8.7.1, the Nash and Pareto solutions will be presented and analysed. The two-stage procedure, extensively set out in section 8.3, will be followed. Hence, we start the control experiments by computing the expectations of the optimal target paths, using the certainty equivalence result (see Appendix 5D). In this first stage the values of the weighting matrices Q_i and R_i, i = NL, G are to be determined. Subsequently, one more experiment is performed: a state model affected by the noise term $\bar{M}v(t+1)$, see (8.4), is considered, and the covariances $\Lambda(t)$ for the target variables will be computed (the expression for $\Lambda(t)$ will be restated in section 8.7.3).

In figures 8.3 and 8.4 we will display the Nash solution for target and instrument variables respectively. The optimal paths will be confronted with the corresponding desired paths for the target variables and for the instrument variables, when G and NL both use policy II. The actual values of the optimal paths will be given later on, in table 8.10.

Conclusions for the Nash solution

The following remarks on the behaviour of the optimal target paths can be made, as examples of economic arguments.

1. The desired values for purchasing power in NL and G are overshot, for all time t of the planning period. This leads to setting the corresponding weights in the Q_i-matrices, i = NL, G, equal to zero. The behaviour of this target variable can be explained from the properties of the solution: all instrument variables are used for demand stimulation, and the most sensitive target variable in this respect is purchasing power (see table 8.5).
2. The optimal path for labour productivity performs satisfactorily: the desired path is tracked closely. Clearly the fact that the weights on purchasing power are zero, increases the possibility of reaching the desired path for labour productivity (no trade-off between these two variables has to be made).
3. The desired path for unemployment is difficult to reach in the case of G, and more easily in the case of NL. This is (partly) caused by the presence of TS(NL): tax reduction in NL reduces unemployment.
4. The optimal target paths for inflation move towards their desired paths, both in NL and G. The large gap in NL between simulated and desired values is reduced by an excessive use of TS(NL). For this reason, unemployment in NL behaves satisfactorily.

From the control results it follows that the optimal path for demand has a minimum in 1986. This holds particularly for NL, as can also be deduced from the labour productivity e2-Emps (figure 8.4). This drop in demand is caused by international trade fluctuations. The uncontrollable exogenous variables $Mg\$_{G,R}$, $Mg\$_{NL,R}$ and $Pxg\$_R$ are used for modelling trade (prices) with the rest-of-the-world.

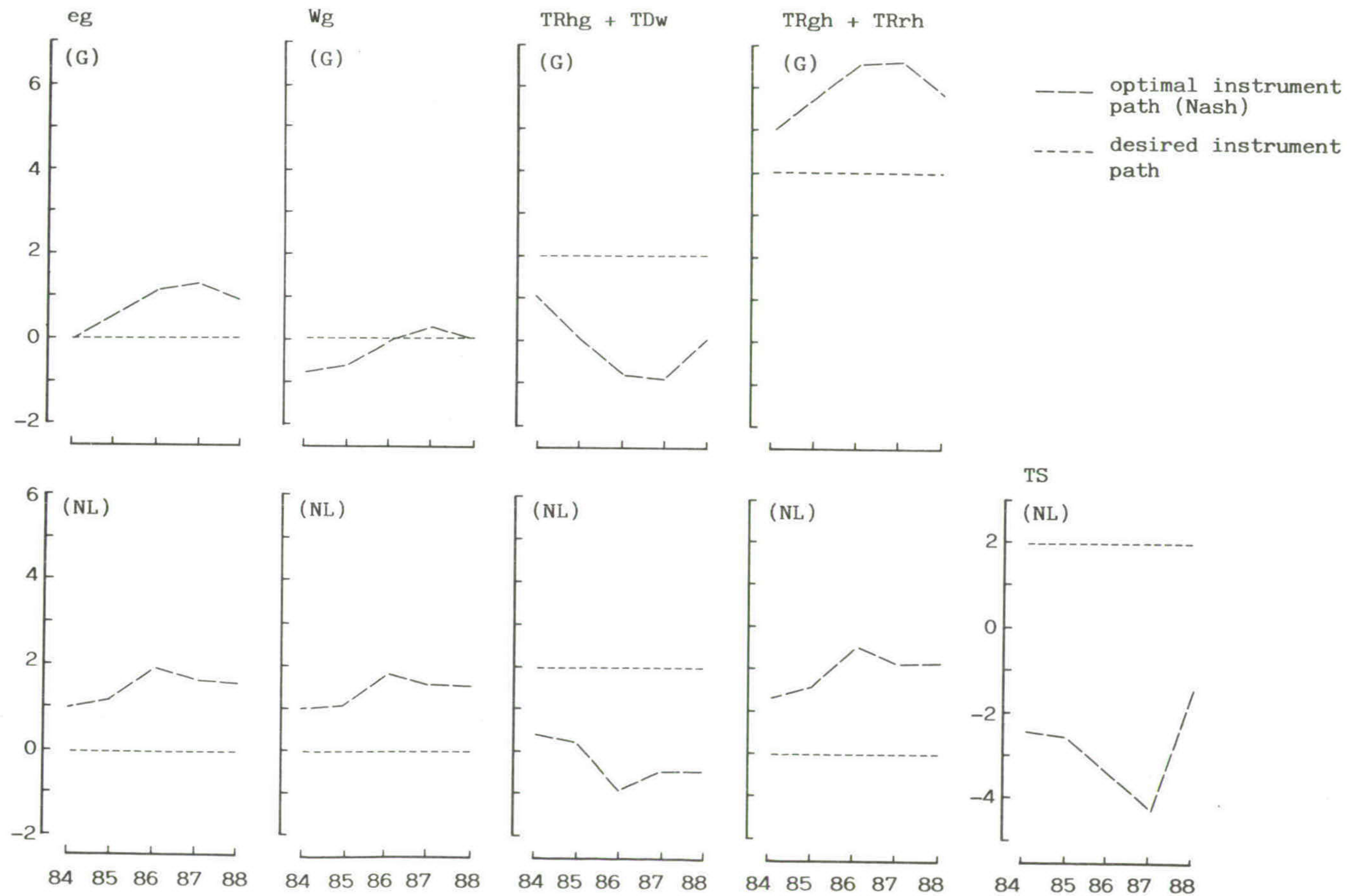


Figure 8.4. The Nash solution: desired instrument path and optimal instrument path.

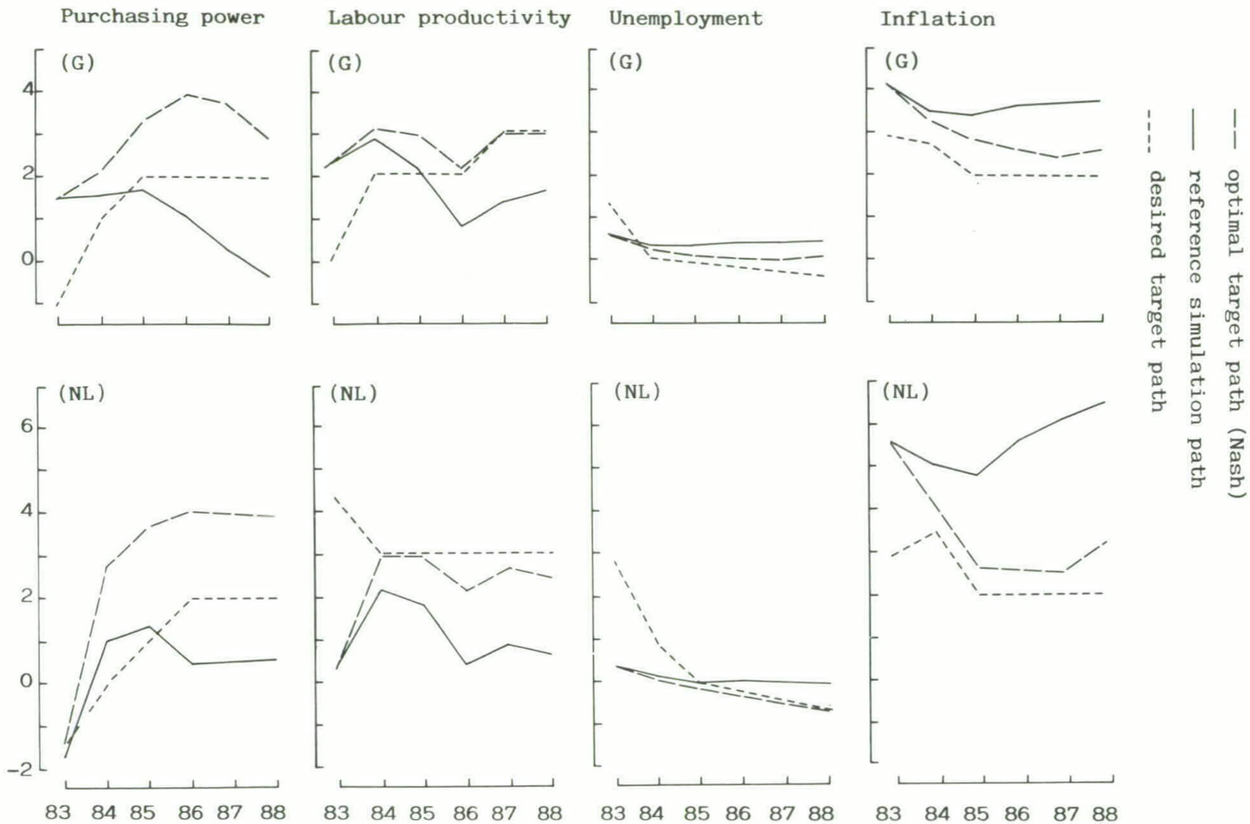


Figure 8.3. The Nash solution: desired target path, reference simulation path and optimal target path.

The anticipated path of these variables is affected by the anticipated drop of MR in 1986 (see table 8.22: -10% in NL and -9% in G). Especially in NL it appears that this reduction in demand is met by additional control effort in 1986: all Dutch instrument variables, except TS, have an extreme in 1986. A similar pattern can be recognized in G, where the control effort in 1986 and 1987 are about equal.

A related remark on the construction of the model is in order here. In a six-countries-model (the original version of Interplay), it is convenient to transform trade variables into US dollars, if compatibility requires this. However, for a G-NL model, the use of a local currency is more natural and advantageous. The exchange rate between the currencies of NL and G is very stable, in contrast to the volatile and unpredictable behaviour of the US dollar. This makes the trade variables $Mg\$_{G,R}$, $Mg\$_{NL,R}$ and $Pxg\$_{R}$ very hard to predict, while the impact of these variables on demand cannot be ignored. Apparently, the first-order approximation (see section 8.2.2) does not hold here. In addition, the most recent observations for these variables lag behind the observations of Dutch and German variables for several years.

Concerning the behaviour of the instrument variables, we observe that all instrument variables are used for the stimulation of demand. The exception is Wg(G), which can be explained from the positive and negative multipliers of Wg(G), see table 8.5. The variable TS(NL) is used mainly for restoration of the inflation rate. It has been argued that a suitable adjustment of the model (section 8.6) can cure this excessive use of TS(NL). Finally we notice that the deviations from the desired instrument paths are similar in both countries, in a qualitative sense. In a quantitative sense, the use of the instrument variables is stronger in NL than in G, since, on average, the multipliers in NL are smaller than in G.

In the following tables we will present the actual values for the Nash and Pareto solutions. Since the Nash and Pareto solutions do not differ greatly, the Pareto solution will not be displayed graphically. Instead, we will compute the difference between Nash and Pareto solutions for several values of α.

Table 8.10. The Nash solution

Optimal instrument paths (u^*_G, u^*_{NL})

	Germany				Netherlands				
year	eg	Wg	TRhg+ TDw	TRgh+ TRrh	eg	Wg	TRhg+ TDw	TRgh+ TRrh	TS
1984	-.03	-.72	1.03	4.87	1.08	.99	.41	5.36	-2.40
1985	.49	-.63	.00	5.80	1.25	1.13	.18	5.56	-2.59
1986	1.09	-.00	-.80	6.53	1.97	1.82	-.93	6.51	-3.41
1987	1.23	.29	-.93	6.64	1.69	1.55	-.49	6.13	-4.32
1988	.93	.04	.03	5.78	1.64	1.55	-.49	6.13	-1.36

Optimal target paths (z^*_G, z^*_{NL})

	Germany				Netherlands			
year	purch. power	labour prod.	unempl. ment	infla- tion	purch. power	labour prod.	unempl. ment	inflation
1983	1.52	2.33	.62	4.20	-1.82	.33	.41	5.59
1984	2.12	3.11	.25	3.31	2.77	2.93	.08	4.15
1985	3.29	2.89	.06	2.84	3.72	2.91	-.16	2.62
1986	3.90	2.10	.02	2.62	4.04	2.14	-.33	2.52
1987	3.72	2.90	-.04	2.38	3.96	2.65	-.52	2.52
1988	2.87	3.00	.04	2.61	3.93	2.36	-.60	3.18

Table 8.11. The Pareto solution, $\alpha = .5$.

Optimal instrument paths (u_G^*, u_{NL}^*)

Germany					Netherlands				
year	eg	Wg	TRhg+ TDw	TRgh+ TRrh	eg	Wg	TRhg+ TDw	TRgh+ TRrh	TS
1984	.15	-.55	.79	5.09	1.05	.96	.46	5.32	-2.38
1985	.67	-.41	-.22	6.00	1.21	1.10	.23	5.51	-2.54
1986	1.36	.30	-1.15	6.83	1.93	1.79	-.87	6.45	-3.34
1987	1.43	.55	-1.17	6.85	1.65	1.52	-.44	6.09	-4.26
1988	1.10	.26	-.21	5.99	1.61	1.52	-.44	6.09	-1.33

Optimal target paths (z_G^*, z_{NL}^*)

Germany					Netherlands			
year	purch. power	labour prod.	unempl. ment	infla- tion	purch. power	labour prod.	unempl. ment	inflation
1983	1.52	2.33	.62	4.20	-1.82	.33	.41	5.59
1984	2.33	3.21	.22	3.26	2.74	2.95	.08	4.14
1985	5.57	3.02	.01	2.74	3.70	2.94	-.16	2.62
1986	4.32	2.30	-.05	2.48	4.03	2.18	-.33	2.51
1987	4.13	3.08	-.10	2.25	3.96	2.69	-.53	2.51
1988	3.29	3.18	-.01	2.50	3.93	2.41	-.60	3.17

Comparison of the Nash and Pareto solutions

Using the results from the multiplier table 8.5, we have claimed that the Nash and Pareto solution are expected to differ only slightly. This claim is motivated by the magnitude of the multipliers between u_i and z_j, $i \neq j$, i,j = NL, G. Now we can make this statement precise, by comparing the difference between the optimal instrument paths and the optimal target paths under the Nash solution and under the Pareto solution (denote this difference by u^*(Nash) - u^*(Pareto) and z^*(Nash) - z^*(Pareto) respectively).

The data in table 8.12 are drawn from tables 8.10 and 8.11. The case of equal weights on the cost functions ($\alpha = \frac{1}{2}$) will be considered first.

Table 8.12. Differences between Nash and Pareto solutions (α = .5).

A. u^*(Nash) - u^*(Pareto)

Germany					Netherlands				
year	eg	Wg	TRhg+ TDw	TRgh+ TRrh	eg	Wg	TRhg+ TDw	TRgh+ TRrh	TS
1984	-.18	-.17	.24	-.22	.03	.03	-.05	.04	-.02
1985	-.18	-.22	.22	-.20	.04	.03	-.05	.05	-.05
1986	-.27	-.30	.35	-.30	.04	.03	-.07	.06	-.07
1987	-.20	-.26	.24	-.21	.04	.03	-.05	.04	-.06
1988	-.17	-.22	.24	-.21	.03	.03	-.05	.04	-.03
average '84-'88	-.20	-.23	.26	-.23	.04	.03	-.05	.05	-.05

B. z^*(Nash) - z^*(Pareto)

Germany					Netherlands			
year	purch. power	labour prod.	unempl. ment	infla- tion	purch. power	labour product	unempl. ment	inflation
1984	-.21	-.10	.03	.05	.03	-.02	.00	.01
1985	-.28	-.13	.05	.10	.02	-.03	.00	.00
1986	-.42	-.20	.07	.14	.01	-.04	.00	.01
1987	-.41	-.18	.06	.13	.00	-.04	.01	.01
1988	-.42	-.18	.05	.11	.00	-.05	.00	.01
average '84-'88	-.35	-.16	.05	.11	.01	-.04	.00	.01

The results of table 8.12 should be interpreted as follows. From part A, in Germany, we observe that 3 instrument values (eg, Wg, TRgh + TRrh) have higher levels under the Pareto solution than under the Nash solution; one instrument variable, the direct taxes and premiums TRhg + TDw, has a lower level. Summarizing, it can be said that demand is stimulated more by u^*(Pareto) than by u^*(Nash). Obviously, this is manifested in the optimal target paths of Germany (see part B). Under the Pareto solution, purchasing power and labour productivity are at higher levels, inflation and unemployment at lower levels than under the Nash solution.

Now let us consider part A, table 8.12 for NL. The situation is quite opposite to the situation in G. Except for TS, demand is stimulated more by u^*(Nash) than by u^*(Pareto), although the difference is almost insignificant. The apparent reason is that German policy is more effective than Dutch policy, and German influence on NL is stronger than vice versa (see table 8.5). This argument may not hold when there is an unequal weighting of target and instrument variables in both cost functions. However, in this particular application the weights Q_{NL} and Q_G are approximately of the same magnitude, and R_{NL} and R_G are both unity matrices (section 8.7.1); hence, the cost functions of both decision makers are similarly structured in this respect. Furthermore, note that indirect taxes TS are slightly higher under the Pareto solution than under the Nash solution. Therefore, the shifts in the Dutch target variables show a similar pattern as the German target variables, except for purchasing power.

A further analysis of the Pareto concept is given for the case $\alpha = .8$ and $\alpha = .2$. In the former case, the German cost function has a weight of .8, in the latter case a weight of .2. The average difference over 1984-1988 will provide the essential information. In all cases, the parameter values of the control problem are as given in section 8.5.

Table 8.13. Average difference between Nash and Pareto solution.

A. α = .8

u^*(Nash) - u^*(Pareto)

	Germany				Netherlands				
	eg	Wg	TRhg+ TDw	TRgh+ TRrh	eg	Wg	TRhg+ TDw	TRgh+ TRrh	TS
average '84-'88	-.05	-.06	.06	-.06	-.04	-.04	.06	-.05	-.06

z^*(Nash) - z^*(Pareto)

	Germany				Netherlands			
	purch. power	labour prod.	unempl. ment	infla- tion	purch. power	labour prod.	unempl. ment	inflation
average '84-'88	-.09	-.04	.02	.03	-.07	-.04	.00	.02

B. α = .2.

u^*(Nash) - u^*(Pareto)

	Germany				Netherlands				
	eg	Wg	TRgh+ TDw	TRgh+ TRrh	eg	Wg	TRhg+ TDw	TRgh+ TRrh	TS
average '84-'88	-.73	-.86	.95	-.85	.17	.15	-.25	.21	-.15

z^*(Nash) - z^*(Pareto)

	Germany				Netherlands			
	purch. power	labour prod.	unempl. ment	infla- tion	purch. power	labour prod.	unempl. ment	inflation
average '84-'88	-1.28	-.58	.22	.39	.10	-.11	.01	.01

For the case α = .8, we observe that for both countries demand is stimulated more for u^*(Pareto) than for u^*(Nash). The effects for both countries are about the same: if we should apply a bargaining concept in order to find the value of α which the decision makers would agree upon (cf. De Zeeuw, 1984, ch. 6), approximately this value of α would be found. Indeed, for this value of α, the target variables of both NL and G under the Pareto solution outmeasure the Nash solution.

For the case $\alpha = .2$, i.e. a rather small weight on the German cost function, there is room for large demand stimulation in G. In a qualitative sense this solution is similar to the one for $\alpha = .5$, only the effects are much greater.

Remark

In general the Pareto solution may differ greatly from the Nash solution for some or all values of α. In such cases it is natural to assume that the Pareto solution requires other values for Q_i and R_i, $i = 1,2$ than the Nash solution. In our approach, we have determined the Q_i, R_i-weights for the Nash solution, and compared this solution with the Pareto solution for the same set of weights. Comparison is possible and meaningful, since for certain values of α ($\alpha \approx .8$) the Nash and Pareto solutions are about the same. This approach is entirely experimental; the value of other experimental approaches (e.g. choose first a certain value of α, then determine Q_i, R_i, repeat this for other values of α) must be established in practice.

8.7.3. Covariance matrices for the target variables

In previous sections we have presented the optimal target paths under the Nash and the Pareto concepts. In this subsection we compute the covariance matrices of the target variables. Only the case of additive noise will be considered.

The theoretical findings of Appendix 5D will be used. For convenience we repeat the main issues.

Omit the uncontrollable exogenous variables in (8.4), then the state-space model is given by

$$x(t+1) = Ax(t) + Bu(t) + \bar{M}v(t+1) \tag{8.7a}$$

$$z(t) = Hx(t) \tag{8.7b}$$

Let the optimal (Nash or Pareto) control for $u := (u_G;\ u_{NL})$ be given by

$$u^*(t) = L(t)x(t) \tag{8.8}$$

The state trajectories are generated by (insert (8.8) into (8.7a))

$$x(t+1) = \bar{A}(t)x(t) + \bar{M}v(t+1) \tag{8.9}$$

with $\bar{A}(t) := A + BL(t)$

Let $\Lambda(t)$ and $\Lambda^z(t)$ be the covariance matrices of $x(t)$ and $z(t)$ respectively, then they satisfy (from (8.7b) and (8.9))

$$\Lambda(t+1) = \bar{A}(t)\Lambda(t)\bar{A}^T(t) + \bar{M}V\bar{M}^T,\ \Lambda(t_0) \tag{8.10}$$

$$\Lambda^z(t) = H\Lambda(t)H^T \tag{8.11}$$

Let $\Lambda^z(t) = (\Lambda^z_{ij}(t))$, $i,j = 1,2,\ldots,r$, then we will present in this section the values for the variances ($\Lambda^z_{ij}(t)$, $t \in [t_0,t_f]$, $i,j = 1,2,\ldots,r$) of the target variables and the correlation coefficients $\rho_{ij}(t) :=$

$$\Lambda^z_{ij}(t)/[\Lambda^z_{ii}(t)\Lambda^z_{jj}(t)]^{\frac{1}{2}},\ i,j = 1,2,\ldots,r,\ t \in [t_0,t_f].$$

The following interpretation may be attached to $\Lambda^z_{ii}(t)$ and $\rho_{ij}(t)$. From the variances $\Lambda^z_{ii}(t)$ we may infer the 2σ-bounds, i.e. $2[\Lambda^z_{ii}(t)]^{\frac{1}{2}}$, which provide a rough estimate of the 95%-confidence interval of the target variable z_i (see Theil, 1971, chapter 2). A caveat is in order here: the 2σ-bounds are in fact not suitable for a simultaneous model and should only be used as a rough approximation.

The correlation coefficients may be interpreted as follows. Let $\rho_{ij}(t) > 0$ (< 0). Then, if $z_i(t)$ turns out to exceed its expected value, $z_j(t)$ will be expected to exceed (fall below) its expected value.

We will present variances and correlation coefficients in detail for the Nash solution and roughly for the Pareto solution. The initial vector $\Lambda(1983)$ is computed from sample covariances of components of the state. Starting with $\Lambda(1983)$ the covariance matrices $\Lambda^z(t)$, $t = 1984,\ldots,1988$ can be computed, using (8.10). To establish convergence, the recursion (8.10) may be recomputed over the planning period starting with the updated initial vector $\Lambda(t_0) := \Lambda(1988)$. The result for 1988 in the second iteration will be presented, as well as the variances over the planning period in the first iteration (see table 8.14).

Table 8.14. Variances and correlation coefficients of the target variables under the Nash solution.

A. Variances of target variables $\Lambda^z_{ii}(t)$, $i = 1,2,\ldots,r$, $t = 1983,\ldots,1988$.

Target variable	Initial vector	First iteration					Second iteration initial vector	
Germany	1983	1984	1985	1986	1987	1988	1983	1988
Purchasing power	16.56	21.68	29.83	34.53	37.71	35.90	35.90	36.65
Labour productivity	13.94	18.48	19.42	20.24	20.87	20.54	20.54	20.68
Unemployment	.69	2.13	2.34	2.43	2.45	2.50	2.50	2.52
Inflation	3.88	10.44	12.25	12.26	12.19	12.38	12.38	12.40
Netherlands	1983	1984	1985	1986	1987	1988	1983	1988
Purchasing power	23.16	4.84	5.55	5.49	5.54	5.71	5.71	5.71
Labour productivity	12.83	4.45	4.48	4.49	4.52	4.53	4.53	4.53
Unemployment	.50	.58	.65	.73	.73	.73	.73	.73
Inflation	9.24	15.43	15.37	15.40	15.45	15.34	15.34	15.34

B. Correlation coefficients of target variables, $\rho_{ij}(t)$, $i,j = 1,2,\ldots,r$, $i \neq j$, $t = 1984,\ldots,1988$.

$z_1 \equiv$ purchasing power (G)	$z_5 \equiv$ purchasing power (NL)
$z_2 \equiv$ labour productivity (G)	$z_6 \equiv$ labour productivity (NL)
$z_3 \equiv$ unemployment (G)	$z_7 \equiv$ unemployment (NL)
$z_4 \equiv$ inflation (G)	$z_8 \equiv$ inflation (NL)

t = 1984

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.78						
z_3	-.41	-.58					
z_4	-.39	-.74	.57				
z_5	.20	.32	-.26	-.30			
z_6	.41	.59	-.51	-.51	.79		
z_7	-.10	-.15	.13	.13	-.19	-.36	
z_8	-.09	-.13	.11	.11	-.08	-.02	.09

t = 1985

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.76						
z_3	-.25	-.48					
z_4	-.32	-.65	.67				
z_5	.17	.30	-.23	-.23			
z_6	.35	.57	-.49	-.50	.77		
z_7	-.10	-.14	.14	.15	.04	-.27	
z_8	-.10	-.13	.13	.14	-.03	-.01	.16

t = 1986

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.77						
z_3	-.15	-.43					
z_4	-.27	-.62	.69				
z_5	.17	.29	-.22	-.23			
z_6	.32	.55	-.49	-.51	.77		
z_7	-.09	-.13	.13	.14	.02	-.28	
z_8	-.09	-.13	.13	.14	-.03	-.01	.14

t = 1987

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.77						
z_3	-.11	-.40					
z_4	-.24	-.60	.69				
z_5	.16	.29	-.22	-.23			
z_6	.31	.55	-.48	-.50	.78		
z_7	-.09	-.13	.13	.14	.00	-.29	
z_8	-.08	-.12	.12	.14	-.02	-.01	.13

t = 1988

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.77						
z_3	-.10	-.39					
z_4	-.23	-.59	.69				
z_5	.16	.29	-.21	-.22			
z_6	.30	.54	-.48	-.50	.77		
z_7	-.09	-.13	.13	.14	.03	-.28	
z_8	-.08	-.12	.12	.13	-.03	-.01	.10

t = 1988, second iteration

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.77						
z_3	-.08	-.39					
z_4	-.22	-.59	.69				
z_5	.16	.29	-.21	-.22			
z_6	.30	.54	-.48	-.50	.77		
z_7	-.08	-.13	.13	.14	.03	-.28	
z_8	-.08	-.12	.12	.13	-.03	-.01	.10

Since the Interplay model displays a very weak interaction between the submodels, we will only investigate the correlation coefficients of the members of the set of German target variables and of the members of the set of Dutch target variables. The results obtained for the two submodels will be compared. The following table draws the essential information from table 8.14B.

Table 8.15. Correlation coefficients compared for G and NL

		Correlation coefficients			
	t =	1984	1988	1984	1988
Target variables		Germany		Netherlands	
purchasing power	- labour product.	.78	.77	.79	.77
	- unemployment	-.41	-.10	-.19	.03
	- inflation	-.39	-.23	-.08	-.03
labour product.	- unemployment	-.58	-.36	-.36	-.28
	- inflation	-.74	-.02	-.02	-.01
unemployment	- inflation	.57	.69	.09	.10

Discussion of the result

The results in tables 8.14 and 8.15 will now be interpreted. In table 8.14A we presented variances of the target variables, which may serve as a measure for confidence intervals of the target variables. The conclusion is that confidence intervals do not grow in time, but will stabilize to a stationary value. Especially for NL convergence is fast: the variances of z_{NL} are settled down in 1988; in the case of G, the recursion (8.10) over the planning period must be restarted twice using updated initial vectors, in order to establish convergence.

The confidence intervals are seen to be rather large (take 2σ-bounds), especially for the German target variables. The difference between German and Dutch target variables is significant. This holds especially for target variables which are composed out of more than one endogenous variable. In that case we must account for the fact that covariances between variables can play a substantial role. This will be illustrated by means of an example for the purchasing power variable Wd-Pcp-Emp. Some notation will be introduced first.

Let Σ_{aa} be the variance of endogenous variable a, Σ_{ab} be the covariance of endogenous variables a and b, and similarly for z, a, b, c. If z = a-b-c, then

$$\Sigma_{zz} = \Sigma_{aa} + \Sigma_{bb} + \Sigma_{cc} + 2(\Sigma_{bc} - \Sigma_{ab} - \Sigma_{ac}).$$

Take z = Wd-Pcp-Emp, a = Wd, b = Pcp, c = Emp, then, from the details of the $\Lambda(t)$-recursion (8.10), we may obtain the following results.

Table 8.16. Variances of purchasing power in G and NL.

z = Wd-Pcp-Emp, a = Wd, b = Pcp, c = Emp				
	Germany		Netherlands	
	t = 1984	1988	t = 1984	1988
Λ_{aa}	37.13	40.35	21.13	21.52
Λ_{bb}	10.44	12.38	15.43	15.34
Λ_{cc}	16.44	19.91	2.24	3.03
Λ_{ab}	-3.71	-4.33	14.27	14.45
Λ_{ac}	16.63	10.92	2.24	2.02
Λ_{bc}	-8.25	-11.77	-.47	-.61
Λ_{zz}	21.68	35.90	4.84	5.71

In table 8.16 we observe that the various components of Λ_{zz} are in general larger in G than in NL. In NL the final result is smaller than in G due to cancellation of the terms. A similar reasoning explains the difference between the variances of the labour productivity variables in NL and G.

From table 8.15 we conclude the following: denote purchasing power and labour productivity as the target variables in group I, and unemployment and inflation as target variables in group II. Then, as a rule, the correlation coefficients between target variables from two separate groups is negative; the correlation coefficients of two target variables within one group is positive. A similar pattern could be found in the multiplier table (table 8.5).

Finally, we notice the most striking difference between G and NL: the correlation coefficient between optimal inflation and optimal unemployment in NL in 1988 is almost zero, but strongly positive in G.

Economic significance can be attached to the correlation coefficients of these two variables. The relation between Pcp and $\Delta\tilde{un}$ is investigated extensively in economic literature, since the seminal article of Phillips on the relation between wage rate and unemployment (see Phillips, 1958). When we designate a positive correlation between inflation and unemployment as a "Phillips-curve", then such a phenomenon can be found in G, but not in NL. In general the analysis of the correlation coefficients can be used to check the theoretical assumptions on the relation between the target or endogenous variables. We will not attempt to derive real-world conclusions from results of table 8.14 and 8.15 here, since the underlying model is only for illustrative purposes.

Remark

The covariance matrices of the target variables under the Pareto concept will not be analysed in detail. It was observed that the variances of the target variables are not very sensitive to the choice of α. The following table, showing variances of the Dutch and German target variables at t = 1988 for five values of α, substantiates this claim.

Table 8.17. Variances of target variables, t = 1988.

Target variables	Pareto solution α = = .95	.80	.50	.20	.05	Nash solution
Purchasing power (G)	35.94	35.90	37.60	35.71	37.23	35.90
Labour prod. (G)	20.53	20.54	20.87	20.62	21.34	20.54
Unemployment (G)	2.50	2.50	2.44	2.47	2.41	2.50
Inflation (G)	12.37	12.36	12.15	12.18	11.80	12.38
Purchasing power (NL)	6.65	5.76	5.53	5.64	5.48	5.71
Labour prod. (NL)	4.58	4.52	4.52	4.55	4.62	4.53
Unemployment (NL)	.73	.73	.73	.73	.72	.73
Inflation (NL)	15.29	15.32	15.45	15.31	15.25	15.34

8.7.4. Digression II. Control experiments in a model with exogenous w(NL)

We will continue the analysis of section 8.4.3, where we have modified the model by taking w(NL) exogenously. In this subsection we will discuss the effect of the adjustment on the control solution. Since we have claimed that extensive analysis of the model properties is required to prepare the control experiment, the analysis presented in previous sections (stability, multipliers, reference simulation, choice of parameters) must be made. The precise impact of the adjustment will be clear from this analysis. However, we will not duplicate this laborious work for the adjusted model, and concentrate on some major topics. Thus, we will present the eigenvalues of the adjusted model, determine new weights for the Dutch cost function, compute optimal instrument paths and expectations and variances of optimal target paths and correlation coefficients of the target variables. Finally we will discuss the results and outline the difference with the original model.

Since the wages in NL are now exogenous variables, the wage-price inflation drift no longer holds. It turns out that the model has lost the complex eigenvalues which induce the business cycle of about 7 years (see section 8.4.2).

Table 8.18. Eigenvalues of system matrix ($|\lambda| > .001$).

w(NL) exogenous		w(NL) endogenous	
1.	.762	1.	.766
2.	.504	2/3.	.413 ± .514i
3.	.494	4.	.505
4.	.402	5.	.490
5.	.315	6.	.299
6.	.188	7.	.160
7/8.	.665 ± .068i	8/9.	.057 ± .047i
9.	-.080	10.	-.079
10.	-.056	11.	-.049
		12.	.0012

Reference simulation paths for the adjusted model have been presented in figure 8.2. We observe that the simulated paths for unemployment and inflation are now closer to their desired values than in the original model. Hence, we expect that less control effort is required for the attainment of the desired paths. Similarly, we expect that more control effort is required for the attainment of the desired paths of purchasing power and labour productivity. The new set of weights that has been determined confirms this. Due to the weak interaction between the submodels, only Q_{NL} needs to be modified. The result is

$$\begin{aligned} \mathrm{diag}(Q_{NL}(t)) &= (10,\ 20,\ 0,\ 0),\ t = 1984,\ 1985 \\ &= (10,\ 30,\ 0,\ 0),\ t = 1986,\ldots,1988. \end{aligned}$$

The resulting paths for target and instrument variables are shown in figures 8.5 and 8.6.

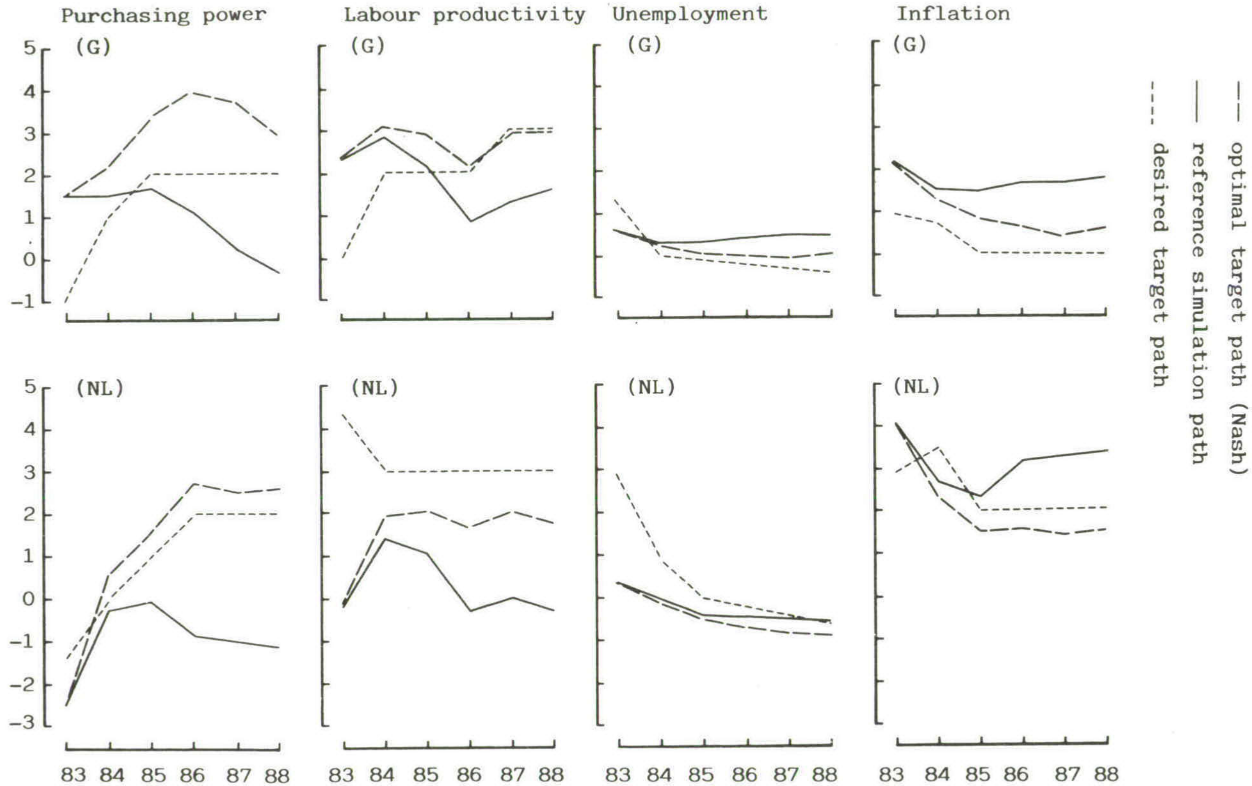


Figure 8.5. The Nash solution: desired target path, reference simulation path and optimal target path. The case of exogenous w_{NL}.

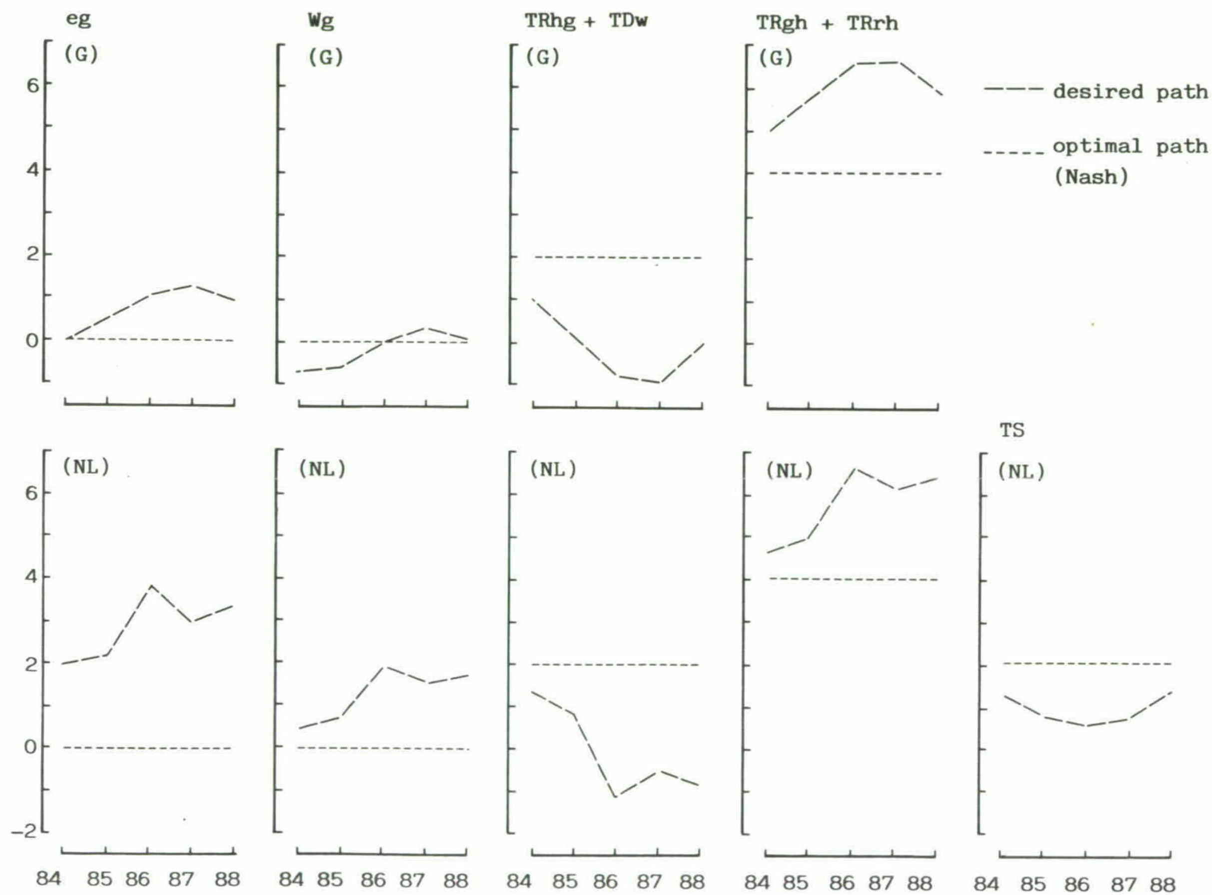


Figure 8.6. The Nash solution: desired instrument path and optimal instrument path.
w(NL) exogenous.

For the Nash solution, variances and correlation coefficients of the target variables have been computed.
Together with values for optimal target and instrument paths these data are summarized in table 8.19.

Table 8.19. The Nash solution for the adjusted model.

(w(NL) exogenous)

A.
Optimal instrument paths (u_G^*, u_{NL}^*)

	Germany				Netherlands				
year	eg	Wg	TRhg+ TDw	TRgh+ TRrh	eg	Wg	TRgh+ TDw	TRgh+ TRrh	TS
1984	-.01	-.72	1.00	4.90	2.00	.48	1.23	4.66	1.27
1985	.50	-.63	-.01	5.81	2.14	.73	.82	5.01	.77
1986	1.09	-.00	-.80	6.52	3.86	1.88	-1.03	6.60	.52
1987	1.23	.29	-.93	6.64	2.97	1.57	-.53	6.16	.67
1988	.93	.05	.03	5.78	3.28	1.75	-.82	6.41	1.35

Optimal target paths (z_G^*, z_{NL}^*)

	Germany				Netherlands			
year	purch. power	labour prod.	unempl ment	infla- tion	purch. power	labour prod.	unempl. ment	inflation
1983	1.52	2.33	.62	4.21	-2.51	-.14	.35	4.07
1984	2.13	3.10	.25	3.32	.63	1.96	-.10	2.30
1985	3.29	2.89	.06	2.84	1.55	2.04	-.51	1.54
1986	3.89	2.10	.01	2.62	2.78	1.69	-.70	1.63
1987	3.72	2.90	-.04	2.38	2.53	2.04	-.83	1.49
1988	2.87	3.00	.04	2.61	2.62	1.76	-.92	1.61

B. Variances of target variables.

Germany	1983	1984	1985	1986	1987	1988
z_1 = purchasing power	16.56	20.97	25.82	27.82	28.96	27.36
z_2 = labour prod.	13.94	5.79	6.38	6.78	7.04	6.72
z_3 = unemployment	.69	.58	.58	.64	.65	.68
z_4 = inflation	3.88	12.38	14.33	14.40	14.35	14.70
Netherlands						
z_5 = purchasing power	23.16	3.32	3.29	3.29	3.29	3.29
z_6 = labour prod.	12.83	2.55	2.52	2.51	2.51	2.51
z_7 = unemployment	.50	.45	.45	.45	.45	.45
z_8 = inflation	9.24	4.16	3.64	3.63	3.63	3.63

C. Correlation coefficients of target variables.

t = 1984

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.88						
z_3	-.51	-.58					
z_4	-.54	-.61	.53				
z_5	.13	.17	-.14	-.16			
z_6	.33	.39	-.35	-.31	.75		
z_7	-.07	-.09	.08	.07	-.20	-.39	
z_8	-.15	-.18	.16	.14	-.93	-.73	.31

t = 1985

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.89						
z_3	-.24	-.39					
z_4	-.43	-.49	.66				
z_5	.13	.17	-.15	-.14			
z_6	.28	.36	-.33	-.31	.76		
z_7	-.07	-.09	.09	.09	-.19	-.40	
z_8	-.15	-.18	.18	.17	-.97	-.77	.31

t = 1986

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.90						
z_3	-.13	-.28					
z_4	-.40	-.47	.64				
z_5	.13	.17	-.14	-.14			
z_6	.26	.34	-.32	-.31	.76		
z_7	-.07	-.09	.08	.09	-.19	-.40	
z_8	-.14	-.17	.17	.18	-.97	-.77	.31

t = 1987

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.90						
z_3	-.10	-.25					
z_4	-.40	-.46	.64				
z_5	.13	.17	-.14	-.14			
z_6	.26	.34	-.32	-.31	.76		
z_7	-.07	-.09	.08	.09	-.19	-.40	
z_8	-.14	-.17	.17	.18	-.97	-.77	.31

t = 1988

	z_1	z_2	z_3	z_4	z_5	z_6	z_7
z_2	.90						
z_3	-.10	-.25					
z_4	-.43	-.48	.65				
z_5	.13	.18	-.13	-.14			
z_6	.26	.34	-.32	-.32	.76		
z_7	-.07	-.09	.08	.09	-.19	-.40	
z_8	-.14	-.17	.17	.18	-.97	-.77	.31

Discussion of the results

First, we will discuss the optimal target paths and the optimal instrument paths under the Nash solution (table 8.19A). Due to the weak interaction between the submodels, the solution for G does not change significantly, compared to the solution of table 8.10. In NL we observe that optimal unemployment and inflation rate fall below their desired paths. Hence, their corresponding weights in the Q_{NL}-matrix are set to zero. The desired path for labour productivity in NL could not be reached, but is approached by 1%; the result for purchasing power in NL is good.

Concerning the instrument variables, we notice that the control effort by eg has increased substantially compared to the original solution (table 8.10). Indeed, this instrument variable is more effective on labour productivity than on purchasing power (see table 8.5). The use of TS is now satisfactory and close to its desired values: no excessive use is required anymore to decrease the inflation rate.

Secondly, we discuss the variances of the target variables (table 8.19B). Apparently the adjustment in the Dutch submodel does have a great effect on the uncertainty by which the target variables will be reached, both in G and in NL. There appears to be a substantial reduction in the variances of all target variables, except Pcp(G). Of course, the covariance of the Dutch wages vanishes; hence, $\Lambda(t_0)$ and $\bar{M}$ change, having less nonzero components. Also the term $\bar{M}V\bar{M}^T$ in (8.10a) changes considerably, see the remark below.

From table 8.19C we observe that the correlation coefficients of inflation and purchasing power in NL, and of inflation and labour productivity in NL change, compared to table 8.11. These two correlation coefficients were about zero in table 8.11, and are now -.97 and -.77 in 1988 respectively. From equation 11, Appendix 8B, we observe that Wd consists of .834(w+Emp), with w now an exogenous variable. When we confront inflation Pcp with purchasing power Wd-Pcp-Emp, it is to be expected that these variables will be strongly negatively correlated. Similarly, but more weakly, we can explain the correlation between Pcp and labour productivity e2-Emps (use equations 1 and 13).

Remark

In the transformation from the structural form (8.2) to the state-space form (8.4), the first step is the transformation of (8.2) to reduced form. The noise term of the reduced form is $(I-A_0)^{-1}M\ v(t) =:$ $\tilde{M}\ v(t)$. The almost completely filled, non-square matrix $\tilde{M}$ is supplemented with zeroes and arises as $\bar{M}$ in the state-space form (see Proposition 5.1). The matrix $\bar{M}V\bar{M}^T$, required in recursion (8.10), displays great differences between the cases of endogenous and exogenous w(NL). The diagonal elements of $\bar{M}V\bar{M}^T$ in both cases will illustrate this. In the first column of table 8.20 we present the endogenous variable of (8.10) of the corresponding state element. The results in table 8.20 explain, among others, the differences in the variances of the target variables in both cases (see tables 8.14A and table 8.19B).

Table 8.20. Diagonal elements of $\bar{M}V\bar{M}^T$.

Corresponding state element	index i	w(NL) endogenous $(\bar{M}V\bar{M}^T)_{ii}$	index i	w(NL) exogenous $(\bar{M}V\bar{M}^T)_{ii}$
cp(G)	1	80.53	1	19.74
$\tilde{ip}$(G)	2	172.01	2	71.55
Δun(G)	3	2.00	3	.44
Emp(G)	4	15.18	4	3.31
mg(G)	5	176.07	5	51.06
Peg(G)	6	5.76	6	1.60
Pcp(G)	7	9.41	7	11.35
Pip(G)	8	19.50	8	9.35
w(G)	9	17.33	9	11.25
cp(NL)	10	3.47	10	3.61
$\tilde{ip}$(NL)	11	48.01	11	40.69
Δun(NL)	12	.41	12	.40
Emp(NL)	13	1.35	13	1.30
mg(NL)	14	21.34	14	17.14
Pxg(NL)	15	1.11	15	1.02
Peg(NL)	16	4.31	16	2.62
Pcp(NL)	17	13.42	17	3.43
Pip(NL)	18	2.69	18	1.41
w(NL)	19	16.08		
mg(G,NL)	20	235.00	19	90.51
mg(NL,G)	21	47.35	20	40.91
Wd(G)	22	26.90	21	12.45
Emps(G)	23	9.17	22	2.00
gvampp(G)	24	44.22	23	12.19
Wd(NL)	25	12.55	24	.90
e2(NL)	26	5.93	25	4.11
Emps(NL)	27	.83	26	.80
e2(G)	28	25.41	27	15.44
W(G)	29	56.30	28	11.76

8.8. Summary and conclusions

In this chapter we have applied LQG-theory for stochastic dynamic games to a macroeconometric policy model. The application consists of preliminary investigations into model properties and control experiments. The major issues will be summarized below.

First, the properties and dynamic behaviour of the model must be analysed. The results lead to the conclusion whether or not the model is adequate for the purpose of policy making. The concepts that may be used are stability (eigenvalues), controllability and observability (multipliers) and goodness-of-fit (performance indices).

Secondly, if the model description is satisfactory, one can proceed to the next stage: the preparation of the control experiment. Values must be attributed to the parameters of the control problem. Great care is needed for this task: e.g. if a quadratic cost function is used, one must take into account that negative and positive deviations from the desired paths are penalized equally, which may be considered unfavourable. Desired paths for instrument and target variables and anticipated paths for uncontrollable exogenous variables must be determined. This may be a difficult task, particularly when the solution of the control problem is sensitive with respect to the choice of these paths.

Thirdly, the control experiment must be performed. The optimal control solution is implemented and yields the optimal instrument paths and the expectation and variance of the optimal target variables. A procedure that minimizes the costs in computation has been suggested.

These three steps have been applied to an existing macroeconometric model based on real-world data. The model is for illustrative purposes only, as the model description is not yet entirely adequate and requires revision. Adjustments of parameter values may also give better results on behalf of goodness-of-fit and predictions. In addition, the effectiveness of the instrument variables is low and the interaction between the decision makers G and NL is not very strong. Hence the game-theoretical application is weak and not very different from a one-player optimal control application. Procedures to determine the parameters of the control problems have been supplied. However, more knowledge and experience must be gathered to find practical procedures determining the weights Q_i, R_i of the cost function, in combination with the Pareto parameter α. Also more experience is required to determine economic forecasts; for instance, a communication process may be set up between model experimenter and actual policy maker in order to agree upon desired and anticipated values for endogenous and exogenous variables. Concerning the uncertainty in the model, we have confined attention to the additive noise case. The covariances of the optimal target variables have been computed, which provide a measure for the confidence intervals of the optimal target paths. Furthermore, they provide correlation coefficients of the optimal target variables; suppose that the underlying economic theory specifies explicitly the direction of the correlation between target or endogenous variables, then the computation of the correlation coefficients can be used as a test.

As an overall conclusion we can state that the application of LQG-theory to stochastic dynamic games is feasible, and leads to tractable and interpretable results. As a prerequisite for application of the control approach, a thorough analysis of the model and its properties is required. In addition, the user must supply (empirical) procedures to determine the parameter values of the control problem.

Very recently an acknowledgement in economic literature has been made, claiming that a large class of macroeconomic policy models should consist of several decision makers (usually the government and the public sector, in view of the Lucas critique, Lucas (1976)). This observation has led to an increase in economic research on the application of game-theoretical concepts in economic modelling (time-consistency, credibility, threats). In applied work, it is evident that tractable results are most easily obtained for linear (linearized) models, afflicted by Gaussian white noise, and quadratic cost functions. Furthermore, it seems evident that a large class of economic models will fit the framework where some of the decision makers have non-shared information. Therefore it is felt that attention will be shifted from the global dynamics, shared information case to models having non-shared information patterns. Some examples of such models have been discussed in this book.

Appendix 8A. List of symbols.

Capital letters denote variables at current prices, lower-case letters denote volumina (except wages w). All variables, except those with a tilde, are annual growth rates. P denotes a price index, Δ a first difference. In the estimation procedure, variables with fractional lags have been used as regressors

$$x_{-\theta} = (1-\theta)x + \theta\, x_{-1},\ \theta \in \{\tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4}\}.$$

The transformation between local currency and US dollars satisfies, for an endogenous or exogenous variable x and the exchange rate MR:

$$x_{\$} = x - MR.$$

Suffix

g	government, goods	r, R	rest-of-the-world
d	disposable	p	private
h	households	c	corporations

Variables

i	investments	e1	total expenditures
c	consumption	e2	total expenditures
$\tilde{un}$	unemployment		minus stocks and invisibles
Em(s)	employment (self-employed)	$\tilde{RL}$	long-term interest rate
H	unit labour costs	MR	exchange rate
gvamp	gross value added at market prices	Fg	budget deficit of government
W	wage income	L1	primary liquidities
NW	non-wage income	L2	secondary liquidities
w	nominal wage income per employee in the private sector	TS	indirect taxes minus subsidies
eg	government expenditures	TR(ab)	transfers (from a to b)
st^*	$\Delta\tilde{st}/\tilde{e1}_{-1}$		
$(xs-ms)^*$	$\Delta(\tilde{xs}-\tilde{ms})/\tilde{e1}_{-1}$	TDw	direct taxes on wages
st	stocks	xg	exports of goods

xs	export of services	mg	import of goods
ms	import of services	Powa	population at working age
Pod	dependent working population	Du	dummy
		DIF	artificial difference variable

Remark. Du69 means that the dummy variable has value 1 in 1969 and zero in all other years of the sample period.

Appendix 8B. Submodels for NL, G and the linking section.

Netherlands

Behaviourial equations

1) $cp = .633(Wd-Pcp) + .175(L1-Pe2)_{-1}$

2) $ip = .160NWc - 7.005\ \Delta\tilde{un} + .944(w-Pe2) - 15.873Du69$

3) $\Delta\tilde{un} = -.43Emp + .256((\Delta Po\tilde{wa}+\Delta E\tilde{mg})/P\tilde{od}_{-1})$

4) $Emp = 1.423 - .193w_{-1} + .177e2 + .059e2_{-1} + .273Emp_{-1}$

5) $mg = -1.297 + 1.599e1 + 5.996Du54$

6) $Pxg = .821Pmg$

7) $Peg = -.316eg + .253(mg-e2) + 1.076Pip$

8) $Pcp = 1.479 + .674H(-\frac{1}{4}) + .057(TS-e2) + .057(TS-e2)_{-1}$

9) $Pip = 1.041 + .523(.495w + .505Pmg) + .174(.495w + .505Pmg)_{-1}$
$-.052(L2 - Pe2) - .017(L2 - Pe2)_{-1}$

10) $w = .869Pcp + .290Pcp_{-1} + .592(e2-Emps) + .197(e2-Emps)_{-1}$
$- 1.304\ \Delta\tilde{un}_{-1}$

Definitional equations

11) $Wd = .834w + .834Emp + .216Wg + .001Wr + .297(TRgh+TRrh)$
$- .347(TDw+TRwg)$

12) $e1 = .950e2 + st^* + (xs-ms)^*$

13) $e2 = .495cp + .162ip + .071eg + .272xg$

14) $Emps = .785Emp + .215Ems$

15) $H(-\frac{1}{4}) = w - .75(e2-Emps) - .25(e2-Emps)_{-1}$

16) $Pe2 = .473Pcp + .155Pip + .073Peg + .299Pxg$

Germany

Behaviourial equations

17) $cp = .908 + .685(Wd-Pcp) + .086(NWhd-Pcp)$

18) $ip = 1.406cp + .005Fg - .262H(-\frac{1}{4}) - .262H(-\frac{1}{4})_{-1}$

19) $\Delta\tilde{un} = .164 - .327Emp$

20) $Emp = .442e2 + .147e2_{-1} - .073w - .219w_{-1}$

21) $mg = 1.775e1 + 10.873Du54$

22) $Pxg = -1.665 + .185w + .185w_{-1} + .403Pmg + .134Pmg_{-1}$

23) $Peg = 1.074Pip - .133eg - 6.999Du58$

24) Pcp $= 1.485 + .507Pcp_{-1} + .337(W\text{-}e2) - .059(W\text{-}e2)_{-1} - .057(W\text{-}e2)_{-2} - .128cp - .063cp_{-1} + .065cp_{-2}$

25) Pip $= .494Pip_{-1} + .822H(-\tfrac{1}{4}) - .406H(-\tfrac{1}{4})_{-1} + .143NWh - .071NWh_{-1}$

26) w $= .372(gvampp\text{-}Emps) + .372(gvampp\text{-}Emps)_{-1} + .566Pip + .302w_{-1}$

Definitional equations

27) Wd $= 1.029W - .292(TRhg+TDw) + .263(TRgh+TRrh)$

28) e1 $= .993e2 + st^{*} + (xs\text{-}ms)^{*}$

29) e2 $= .535cp + .204ip + .104eg + .158xg$

30) Emps $= .777Emp + .223Ems$

31) $H(-\tfrac{1}{4})$ $= w - .75(gvampp\text{-}Emps) - .25(gvampp\text{-}Emps)_{-1}$

32) gvampp $= 1.139e1 - .139mg$

33) NWhd $= 1.210NWh - .192TDNWh - .018TRNWhg$

34) W $= .829w + .829Emp + .165Wg + .006Wr$

Linking section

Behavioural equations

35) $mg_{G,NL} = 1.075mg_{G} + 1.925\ (\tilde{R}L_{G} - \tilde{R}L_{NL})_{-\frac{1}{2}}$

36) $mg_{NL,G} = 1.197mg_{NL} - .321(Pmg\$_{NL,G} - Pmg\$_{NL}) - 1.190\ (\tilde{R}L_{NL} - \tilde{R}l_{G})_{-1}$

Definitional equations

37) $xg_{G} = .097(mg_{NL,G} + Pmg\$_{NL,G}) + .903Mg\$_{R,G} - Pxg\$_{G} + DIF_{G}$

38) $xg_{NL} = .255(mg_{G,NL} + Pmg\$_{G,NL}) + .745Mg\$_{R,NL} - Pxg\$_{NL} + DIF_{NL}$

39) $Pxg\$_{G} = Pxg_{G} - MR_{G}$

40) $Pxg\$_{NL} = Pxg_{NL} - MR_{NL}$

41) $Pmg_{G} = Pmg\$_{G} + MR_{G}$

42) $Pmg_{NL} = Pmg\$_{NL} + MR_{NL}$

43) $Pmg\$_{G} = .036Pxg\$_{NL} + .961Pxg\$_{R} + 10.22Du73 - 17.04Du74 - 10.36Du80$

44) $Pmg\$_{NL} = .181Pxg\$_{G} + .826Pxg\$_{R}$

45) $Pmg\$_{G,NL} = .320Pxg\$_{NL} + .689Pxg\$_{R} + 7.91Du73 - 18.04Du74 - 10.34Du80$

46) $Pmg\$_{NL,G} = .756Pxg\$_{G} + .249Pxg\$_{R}$

Table 8.21

A. Standard deviations of residuals of behaviourial equations.

Equation number	1	2	3	4	5	6	7	8	9	10	17
Standard deviation	1.18	4.53	.30	.61	1.78	1.10	2.27	1.14	1.65	1.99	.91

Equation number	18	19	20	21	22	23	24	25	26	35	36
Standard deviation	3.62	.35	.94	2.76	.99	1.59	.68	1.28	1.17	5.56	4.00

B. Sample standard deviations of endogenous variables.

Endogenous variable	Sample st. deviation	Endogenous variable	Sample st. deviation
cp(G)	2.63	cp(NL)	2.38
ip(G)	6.84	ip(NL)	8.36
$\Delta\tilde{u}n$(G)	.83	$\Delta\tilde{u}n$(NL)	.71
Emp(G)	2.31	Emp(NL)	1.37
mg(G)	6.88	mg(NL)	7.14
Pxg(G)	3.47	Pxg(NL)	6.47
Peg(G)	3.86	Peg(NL)	6.00
Pcp(G)	1.97	Pcp(NL)	3.04
Pip(G)	2.90	Pip(NL)	3.41
w(G)	2.74	w(NL)	3.52
mg(G,NL)	9.17	mg(NL,G)	9.59
Wd(G)	2.71	Wd(NL)	3.47
Emps(G)	1.82	e2(NL)	3.43
gvampp(G)	3.26	Emps(NL)	1.03
W(G)	3.46		
e2(G)	3.31		

Appendix 8C. Data for control experiments.

Estimation period 1953-1980
Simulation period 1980-1983
Planning period 1983-1988

Table 8.22. Desired paths for exogenous variables.

a. Germany

year	$\tilde{RL}$	Fg	MR	NWh	TRNWhg	TDNWh	Wr	Ems	st*	(xs-ms)*
1980	8.5	-10.1	-.83	3.55	6.92	-.31	8.13	-1.89	-.5	.1
1981	10.4	-38.6	24.33	-.54	4.71	-7.87	10.03	-2.11	-1.0	.2
1982	9.0	9.3	7.37	7.80	9.18	-5.31	9.57	-1.02	-.3	.2
1983	7.9	14.3	6.5	5.0	9.09	-3.84	5.0	-.5	.7	.2
1984	7.0	5.0	15.0	5.0	7.0	-1.0	0.0	-.4	-.2	.5
1985	6.0	0.0	0.0	5.0	5.0	2.0	0.0	0.0	-.3	.5
1986	5.5	-5.0	-9.0	4.0	4.0	2.0	0.0	0.0	-.1	.2
1987	5.0	-5.0	-4.0	3.0	3.0	2.0	0.0	0.0	-.1	.2
1988	4.5	0.0	0.0	2.0	2.0	2.0	0.0	0.0	.7	.2

	eg		Wg		TRhg+TDw		TRgh+TRrh	
	policy I	policy II	policy I	policy II	policy I	policy II	policy I	policy II
1980	2.85	2.85	7.06	7.06	10.49	10.49	5.66	5.66
1981	-1.84	-1.84	6.41	6.41	6.86	6.86	8.21	8.21
1982	-4.73	-4.73	2.76	2.76	5.66	5.66	6.26	6.26
1983	-3.0	-3.0	0.0	0.0	2.57	2.57	4.0	6.0
1984	-1.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1985	-1.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1986	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1987	-3.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0
1988	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0

b. Netherlands.

year	exogenous variables									$\frac{\Delta\widetilde{Powa} + \Delta Emg}{\Delta\widetilde{Pod}_{-1}}$
	$\widetilde{RL}$	MR	L1	NWc	Wr	L2	st^{*}	$(xs\text{-}ms)^{*}$	Ems	
1980	10.2	-.90	6.00	-3.32	20.0	.57	-.06	-0.00	-.32	2.15
1981	11.6	25.51	-2.38	8.33	-8.3	17.43	-1.60	.65	-.96	2.81
1982	10.1	7.01	9.85	9.36	90.9	4.72	.69	.07	-1.45	2.34
1983	8.6	6.73	10.14	9.88	-19.1	10.66	-.15	-.42	-.66	2.17
1984	8.0	15.80	4.0	9.0	10.0	10.0	1.75	.16	-.17	2.0
1985	6.8	0.0	3.0	7.0	0.0	9.0	-.5	.16	-.10	1.7
1986	6.4	-10.0	3.0	5.0	0.0	8.0	-.1	.16	-.10	1.3
1987	5.7	-5.0	3.0	3.0	0.0	7.0	.2	.16	-.10	1.0
1988	5.2	0.0	3.0	1.0	0.0	6.0	.1	.16	-.10	.70

	eg		Wg		TRhg+TDw		TRgh+TRrh		TS	
	policy I	policy II	policy I	policy II	policy I	policy II	policy I	policy II	policy I	policy II
1980	0.00	0.00	5.00	5.00	8.86	8.86	8.68	8.68	7.45	7.45
1981	-.41	-.41	2.13	2.13	4.18	4.18	10.86	10.86	2.75	2.75
1982	-3.25	-3.25	3.64	3.64	7.41	7.41	10.06	10.06	-.70	-.70
1983	1.70	1.70	.69	.69	8.83	8.83	5.31	5.31	3.61	3.61
1984	0.0	0.0	-2.0	0.0	2.0	2.0	2.0	4.0	0.0	2.0
1985	-1.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1986	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1987	-3.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0
1988	-2.0	0.0	-2.0	0.0	0.0	2.0	2.0	4.0	0.0	2.0

c. Linking section

year	exogenous variables		
	$\tilde{M}g\$_{R,NL}$	$\tilde{M}g\$_{R,G}$	$Pxg\$_R$
1980	17.1	12.6	23.6
1981	-14.0	-7.0	-11.0
1982	0.0	1.0	-3.5
1983	-1.0	0.0	0.0
1984	-5.0	-4.0	-8.0
1985	7.0	6.0	3.0
1986	15.0	14.0	15.0
1987	10.0	9.0	8.0
1988	3.0	2.0	3.0

Table 8.23. Desired paths for target variables.

a. Germany.

year	target variables			
	purchasing power	labour productivity	unemploy-ment	inflation
1980	-.27	.77	0.00	5.38
1981	-.29	.99	1.55	6.02
1982	-.30	-3.09	1.99	5.27
1983	-1.0	0.0	1.30	2.96
1984	1.0	2.0	0.00	2.75
1985	2.0	2.0	-.10	2.0
1986	2.0	2.0	-.20	2.0
1987	2.0	3.0	-.30	2.0
1988	2.0	3.0	-.40	2.0

b. Netherlands.

year	target variables			
	purchasing power	labour productivity	unemploy-ment	inflation
1980	-1.96	-.62	.86	7.34
1981	.78	-.70	3.22	6.29
1982	2.05	1.60	3.60	5.43
1983	-1.44	4.33	2.86	2.95
1984	0.0	3.0	0.91	3.50
1985	1.0	3.0	0.0	2.0
1986	2.0	3.0	-0.2	2.0
1987	2.0	3.0	-0.4	2.0
1988	2.0	3.0	-0.6	2.0

References

AOKI, M. (1976), Optimal control and system theory in dynamic economic analysis, North-Holland, Amsterdam.

AOKI, M. (1984), Notes on economic time series analysis: system theoretic perspectives, Lecture Notes in Economics and Mathematical Systems 220, Springer-Verlag, Berlin.

ASH, R.B. (1972), Real analysis and probability, Academic Press, New York.

ÅSTROM, K.J. (1970), Introduction to stochastic control theory, Academic Press, New York.

ATHANS, M. (1968), The matrix minimum principle, Information and Control, vol. 11, pp. 592-606.

BAGCHI, A. (1984), Stackelberg differential games in economic models, Lecture Notes in Control and Information Sciences 64, Springer-Verlag, Berlin

BAGCHI, A. and G.J. OLSDER (1981), Linear-quadratic stochastic persuit-evasion games, Applied Mathematics and Optimization, vol. 7, pp. 95-123.

BAR-SHALOM, Y. and E. TSE (1974), Dual effect, certainty equivalence and separation in stochastic control, IEEE Trans. Automatic Control, vol. AC-19, pp. 494-500.

BARTEN, A.P., G. D'ALCANTARA and G.J. CARRIN (1976), Comet, a medium-term macro-economic model for the European Economic Community, European Economic Review, vol. 7, pp. 63-115.

BAŞAR, T. (1978), Two-criteria LQG-decision problems with one-step delay observation sharing pattern, Information and Control, vol. 38, pp. 21-50.

BAŞAR, T. and G.J. OLSDER (1982), Dynamic noncooperative game theory, Academic Press, New York.

BERTSEKAS, D.P. (1976), Dynamic programming and stochastic control, Academic Press, New York.

BISMUT, J.M. (1973), An example of interaction between information and control: the transparancy of a game, IEEE Trans. Automatic Control, vol. AC-18, pp. 518-522.

BORKAR, V. and P. VARAIYA (1982), Asymptotic agreement in distributed estimation, IEEE Trans. Automatic Control, vol. AC-27, pp. 650-655.

BOX, G.E.P. and G.M. JENKINS (1976), Time series analysis: forecasting and control (revised edition), Holden-Day, San Francisco.

BRAINARD, W. (1967), Uncertainty and the effectiveness of policy, American Economic Review, vol. 57, pp. 411-425.

BUITER, W.H. (1983), Expectations and control theory, Economie Appliquée, vol. XXXVI, pp. 129-156.

CALLIER, F.M. and C.A. DESOER (1982), Multivariable feedback systems, Springer-Verlag, Berlin.

CHEN, C.T. (1984), Linear system theory and design, Holt, Rinehart and Winston, New York.

CHOW, G.C. (1975), Analysis and control of dynamic economic systems, John Wiley, New York.

CYERT, R.M. and M.H. DEGROOT (1970), Bayesian analysis and duopoly theory, Journal of Political Economy, vol. 78, pp. 1168-1184.

DAMME, E.E.C. van (1983), Refinements of the Nash equilibrium concept, Lecture Notes in Economics and Mathematical Systems 219, Springer-Verlag, Berlin.

DAVISON, E.J. (1976), Decentralized stabilization and regulation in large multivariable systems, in: Directions in large-scale systems, Y.C. Ho and S.K. Mitter (eds.), Phenum Press, pp. 303-323.

DAVISON, E.J. and N.K. TRIPATHI (1978), The optimal decentralized control of a large power system: load and frequency control, IEEE Trans. Automatic Control, vol. AC-23, pp. 312-325.

DAVISON, E.J. and U. OZGUNER (1983), Decentralized control of traffic networks, IEEE Trans. Automatic Control, vol. AC-28, pp. 677-688.

DESOER, C.A. (1970), Notes for a second course on linear systems, Van Nostrand Reinhold Cie, New York.

DRENICK, R.F. (1981), Large-scale system theory in the 1980's, Large Scale Systems, vol. 2, pp. 29-43.

FAMA, E. (1976), Foundations of finance, Basic Books, New York.

FLEMING, W.H. and R.W. RISHEL (1975), Deterministic and stochastic optimal control, Springer-Verlag, Berlin.

FRANK, P.M. (1978), Introduction to system sensitivity theory, Academic Press, New York.

FRIEDMAN, B.M. (1975), Economic stabilization policy: methods in optimization, North-Holland, Amsterdam.

FRIEDMAN, J.W. (1977), Oligopoly and the theory of games, North-Holland, Amsterdam.

GARBADE, K.D. (1975), Discretionary control of aggregate economic activity, Lexington, Heath.

GOODWIN, G.C. and R.L. PAYNE (1977), Dynamic system identification: experiment design and data analysis, Academic Press, New York.

HAAS, H. (1981), Optimale Steuerung unter Berücksichtigung mehrerer Entscheidungsträger, Ph-D thesis, Bielefeld University (in German).

HAIMES, Y.Y. (1977), Hierarchical analysis of water resources systems, McGraw-Hill Book Cie, New York.

HARSANYI, J.C. (1968), Games with incomplete information played by "Bayesian" players, part I, II, III, Management Science, vol. 14, pp. 159-182, 320-334, 486-502.

HAUTUS, M.L.J. (1970), Stabilization, controllability and observability of linear autonomous systems, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen series A, vol. 73, pp. 448-455.

HO, Y.C. (1970), Differential games, dynamic optimization and generalized control theory, Journal of Optimization Theory and Applications, vol. 6, pp. 179-209

HO, Y.C., M.P. KASTNER and E. WONG (1978), Teams, signaling and information theory, IEEE Trans. Automatic Control, vol. AC-23, pp. 305-311.

HOLT, C.C. (1962), Linear decision rules for economic stabilization and growth, The Quaterly Journal of Economics, vol. 76, pp. 20-45.

INTRILIGATOR, M.D. (1978), Econometric models, techniques and applications, North-Holland, Amsterdam.

JAMSHIDI, M. (1983), Large-scale systems. Modeling and control, North-Holland, Amsterdam.

JAZWINSKI, A.H. (1970), Stochastic processes and filtering theory, Academic Press, New York.

JOHANSEN, L. (1977), Lectures on macroeconomic planning. Part I: General Aspects; Part II: Centralization, decentralization, planning under uncertainty, North-Holland, Amsterdam.

KAILATH, T. (1980), Linear Systems, Prentice-Hall Inc., Englewood Cliffs, N.J.

KALMAN, R.E. (1960), A new approach to linear filtering and prediction problems, ASME Transactions, part D, vol. 82, pp. 95-108.

KALMAN, R.E., P.L. FALB and M.A. ARBIB (1969), Topics in mathematical system theory, McGraw-Hill Book Cie, New York.

KENDRICK, D. (1981), Stochastic control for economic models, McGraw-Hill Book Cie, New York.

KHALIL, H.K. and P.V. KOKOTOVIC (1978), Control strategies for decision makers using different models of the same system, IEEE Trans. Automatic Control, vol. AC-23, pp. 289-297.

KOUTSOYIANNIS, A. (1979), Modern microeconomics (second edition), The Macmillan Press, London.

KUMAR, P.R. and J.H. van SCHUPPEN (1981), On the optimal control of stochastic systems with an exponential-of-integral performance index, Journal of Mathematical Analysis and Applications, vol. 80, pp. 312-332.

KURTARAN, B.Z. and R. SIVAN (1974), Linear-quadratic-Gaussian control with one-step-delay sharing pattern, IEEE Trans. Automatic Control, vol. AC-19, pp. 571-574.

KWAKERNAAK, H. and R. SIVAN (1972), Linear optimal control systems, John Wiley, New York.

LJUNG, L. and T. SODERSTROM (1983), Theory and practice of recursive identification, The MIT Press, Cambridge, Mass.

LUCAS, R.E. (1976), Econometric policy evaluation: a critique, in: The Phillips curve and labor markets, K. Brunner, A.H. Meltzer (eds.), Carnegie-Rochester Conference Series on Public Policy, North-Holland, Amsterdam, pp. 19-64.

LUCAS, R.E. and T.J. SARGENT, eds. (1981), Rational expectations and econometric practice, Minneapolis, University of Minnesota Press.

LUCE, R.D. and H. RAIFFA (1957), Games and decisions, John Wiley, New York.

LUENBERGER, D.G. (1969), Optimization by vector space methods, John Wiley, New York.

MADDALA, G.S. (1979), Econometrics, McGraw-Hill Book Cie, New York.

MARSCHAK, J. and R. RADNER (1972), Economic theory of teams, Yale University Press.

MOORE, B.C. (1981), Principal component analysis in linear systems: controllability, observability and model reduction, IEEE Trans. Automatic Control, vol. AC-26, pp. 17-32.

MURATA, Y. (1982), Optimal control for linear discrete-time economic systems, Springer-Verlag, Berlin.

MYERSON, R.B. (1978), Refinements of the Nash equilibrium concept, Int. Journal of Game Theory, vol. 7, pp. 73-80.

ORTEGA, J.M. and W.C. RHEINBOLDT (1970), Iterative solution of nonlinear equations in several variables, Academic Press, New York.

OWEN, G. (1982), Game theory (second edition), Academic Press, New York.

PADULO, L. and M.A. ARBIB (1974), System theory, Hemisphere Pub. Corp., Washington.

PAPAVASSILOPOULOS, G.P. (1982), On the linear-quadratic-Gaussian Nash game with one-step delay observation sharing pattern, IEEE Trans. Automatic Control, vol. AC-27, pp. 1065-1071.

PAPAVASSILOPOULOS, G.P. (1985), Adaptive dynamic Nash games: an example, 7th SEDC Conf. on Economic Dynamics and Control, London, June 1985.

PHILLIPS, A.W. (1958), The relation between unemployment and the rate of change of money wage rates in the United Kingdom, 1861-1957, Economica, vol. 25, pp. 283-299.

PINDYCK, R.S. (1973), Optimal planning for economic stabilization: the application of control theory to stabilization policy, North-Holland, Amsterdam.

PINDYCK, R.S. (1977), Optimal economic stabilization policies under decentralized control and conflicting objectives, IEEE Trans. Automatic Control, vol. AC-22, pp. 517-530.

PITCHFORD, J.D. and S.J. TURNOVSKY, eds. (1977), Applications of control theory to economic analysis, North-Holland, Amsterdam.

PLASMANS, J.E.J. (1980), Interplay. A linked model for economic policy in the EEC, Working paper nr. 80.101, Tilburg University.

PLASMANS, J.E.J. (1981), Interplay: a linked model for economic policy in the EEC, in: Dynamic modelling and control of national economies (IFAC), J.M.L. Janssen (ed.), Pergamon Press, Oxford, pp. 43-56.

PLASMANS, J.E.J. (1984), Specification and estimation of the linkage block of Interplay II (1953-1980), Working paper nr. 84.17, Tilburg University.

RUSTEM, B. (1983), Rival models, min-max problems and algorithms, preprints 4th SEDC Conf. on Economic Dynamics and Control, Pergamon Press, Oxford, pp. 23-29.

RUSTEM, B. (1985), Methods for the simultaneous use of multiple models in macroeconomic policy optimization, 7th SEDC Conf. on Economic Dynamics and Control, London, June 1985.

SAFONOV, M.G. (1980), Stability and robustness of multivariable feedback systems, The MIT Press, Cambridge, Mass.

SANDELL, N.R., P. VARAIYA, M. ATHANS and M.G. SAFONOV (1978), Survey of decentralized control methods for large scale systems, IEEE Trans. Automatic Control, vol. AC-23, pp. 108-128.

SANDERS, C.W., E.C. TACKER and T.D. LINTON (1974), A new class of decentralized filters for interconnected systems, IEEE Trans. Automatic Control, vol. AC-19, pp. 259-262.

SIMON, H.A. (1956), Dynamic programming under uncertainty with a quadratic criterion function, Econometrica, vol. 24, pp. 74-81.

SINGH, M.G. and A. TITLI (1978), Systems decomposition, optimization and control, Pergamon Press, Oxford.

STRANG, G. (1980), Linear algebra and its applications (second edition), Academic Press, New York.

STRIEBEL, C. (1975), Optimal control of discrete time stochastic systems, Lecture Notes in Economics and Mathematical Systems 110, Springer-Verlag, Berlin.

THEIL, H. (1957), A note on certainty equivalence in dynamic planning, Econometrica, vol. 25, pp. 346-349.

THEIL, H. (1964), Optimal decision rules for government and industry, North-Holland, Amsterdam.

THEIL, H. (1971), Principles of econometrics, John Wiley, New York.

TINBERGEN, J. (1956), Economic policy: principles and design (4th revised printing, 1967), North-Holland, Amsterdam.

TSITSIKLIS, J.N. and M. ATHANS (1984), Convergence and asymptotic agreement in distributed decision problems, IEEE Trans. Automatic Control, vol. AC-29, pp. 42-50.

VARAIYA, P. (1985), Consensus in distributed estimation, in: Systems and Optimization, Proc. of the Twente Workshop, April 1984, A. Bagchi, H.Th. Jongen (eds.), Lecture Notes in Control and Information Sciences 66, Springer-Verlag, Berlin, pp. 182-191.

WAELBROECK, J. ed. (1976), The models of the project LINK, North-Holland, Amsterdam.

WILLMAN, W.W. (1969), Formal solutions for a class of stochastic persuit-evasion games, IEEE Trans. Automatic Control, vol. AC-14, pp. 504-509.

WITSENHAUSEN, H.S. (1968), A counterexample in stochastic optimum control, SIAM J. Control, vol. 6, pp. 131-147.

WONHAM, W.M. (1968), On the separation theorem of stochastic control, SIAM J. Control, vol. 6, pp. 312-326.

ZEEUW, A.J. de (1984), Difference games and linked econometric policy models, Ph.D. Thesis, Tilburg University.

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Samenvatting

De motivatie voor het onderzoek is een bepaald type economisch planningsprobleem. In algemene termen geformuleerd stellen we ons de vraag: hoe dient een economisch agent zijn instrumenten in te zetten, zodat hij de toestand van de economie kan beïnvloeden in een door hem gewenste richting. Voor dit probleem is een kwantitatieve benadering gekozen, in navolging van J. Tinbergen. Het resultaat is een modelmatige methode om de economie te besturen, hetgeen uitgangspunt voor of ondersteuning van beleid kan zijn.

De essentie van de beschrijving van de economie is de relatie tussen de instrumentvariabelen van de agent en de doelvariabelen, die hij (zij) pobeert te beïnvloeden. Deze relatie dient de belangrijkste aspecten van de economie weer te geven. In dit proefschrift zijn de volgende aannames gemaakt.

In de eerste plaats beschouwen we een economie met verschillende agenten. Elke agent heeft een eigen kostenfunctie, terwijl zijn belangen kunnen conflicteren met die van de andere spelers.

In de tweede plaats beschouwen we een dynamisch model en veronderstellen we een planningsperiode die zich over verschillende tijdstippen uitstrekt. Een voorbeeld is de midden-lange-termijnplanning met behulp van een macro-econometrisch model.

In de derde plaats dient de onzekerheid, die het economisch beslissingsproces kenmerkt, in rekening worden genomen. De modelkeuze weerspiegelt deze eis: een klasse van stochastische modellen dient te worden geïntroduceerd als adequate beschrijving van de economie.

Ten slotte worden beslissingen geformuleerd met behulp van de optimale besturingstheorie. Dat wil zeggen, een agent minimaliseert een kostenfunctie onder de beperking van een stochastisch dynamisch model. Gezien de opzet met verschillende agenten, spreken we van een stochastisch dynamisch spel.

Het doel van het onderzoek is het economisch systeem op adequate wijze te beschrijven als een stochastisch dynamisch spel. Vervolgens dient voor het wiskundige model een optimaal besturingsprobleem geformuleerd te worden, zodanig dat praktisch hanteerbare oplossingen beschikbaar komen. En er dient een evaluatie van de methode plaats te vinden,

aan de hand van een geschat en geïmplementeerd econometrisch model. Hiervoor is het Interplay model beschikbaar, een stelsel gekoppelde macro-econometrische modellen van enkele landen van de Europese Gemeenschap.

Het is duidelijk dat slechts in bepaalde gevallen resultaten kunnen worden bereikt die een succesvolle toepassing mogelijk maken. De volgende aannames bleken geschikte aanknopingspunten voor nadere analyse te zijn.

- Het economisch systeem wordt gerepresenteerd met behulp van een econometrisch model in structurele vorm, waarin de onzekerheid wordt gerepresenteerd door additieve witte ruis. Ook de waarnemingen van de spelers kunnen met additieve witte ruis behept zijn.
- Het stochastisch dynamisch spel valt binnen de klasse van LKG-modellen: dat wil zeggen, het model is lineair, de kostenfuncties zijn kwadratisch en de verstoringen zijn Gaussisch (en wit).
- De interactie tussen de spelers wordt vastgelegd door een speltheoretisch concept: Nash (als competatief concept), Pareto (als coöperatief concept) of Stackelberg (als hiërarchisch concept).

Bij de analyse van een stochastisch dynamisch spel staat het begrip informatie centraal. Twee soorten van informatie worden onderscheiden. Enerzijds beschikt de agent over structurele informatie (de structuur en de parameters van het model), anderzijds over zogenaamde on-line informatie, dat wil zeggen waarnemingen over economische variabelen. De agenten in het spel kunnen hun informatie met elkaar delen of niet. In het geval van een spel met 2 agenten, volgt een classificatie in vier verschillende gevallen. Voor elk van deze vier gevallen is een passende economische interpretatie te geven. Omgekeerd kan een economisch fenomeen, na analyse van het bijbehorende informatiepatroon (dat wil zeggen, welke agent weet wat), binnen deze classificatie gemodelleerd worden. Drie van voornoemde gevallen zijn nader geanalyseerd, toegespitst op de formulering en oplossing van het corresponderende optimale besturingsprobleem.

Het eerste geval, waarin alle agenten identieke informatie hebben, valt binnen het klassieke LKG-kader. Het besturingsprobleem wordt opgelost, geïmplementeerd en toegepast in geval van het Nash concept en het Pareto concept. Niet alleen worden de optimale paden van de doelvariabelen berekend, maar ook de bijbehorende betrouwbaarheidsintervallen.

Deze intervallen zijn een indicatie voor de nauwkeurigheid van de berekende resultaten voor de planningsperiode.

In het tweede geval wordt niet alle informatie door de agenten gedeeld. Zij kennen beiden hetzelfde model, maar doen verschillende waarnemingen met betrekking tot de toestand van het economisch systeem. Het optimale besturingsprobleem is alleen oplosbaar indien vereenvoudigingen worden aangebracht. In het bijzonder is onderzocht of de klasse van toegelaten strategieën kan worden ingeperkt door middel van zogenaamde compensatoren. Gegeven de structuur van de compensator kunnen optimale strategieën worden afgeleid; voor de implementatie zijn numerieke oplossingsmethoden noodzakelijk.

In het derde geval hebben de agenten zowel verschillende structurele als on-line informatie. Het economisch systeem is op te vatten als een verzameling lokale subsystemen in een netwerk van gespecificeerde informatiestromen. Alleen voor modellen met een speciale structuur is het mogelijk besturingsproblemen te formuleren en op te lossen (of te benaderen). Vaak gebeurt dit op heuristieke wijze. Het effect van uitwisseling van informatie is onderzocht voor een klasse van modellen met specifieke structuur (namelijk lokale systemen zonder besturing).

De theorie is toegepast op een macro-econometrisch model. Alleen het geval waarin alle agenten dezelfde informatie bezitten (het LKG-geval) is uitgewerkt. Er is gebruik gemaakt van een versie van het Interplay model, gebaseerd op economische tijdreeksen die lopen van 1950 tot 1980. De bedoelde versie omvat twee agenten: de overheden van Nederland en de Bondsrepubliek Duitsland.

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